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CONDENSATS DE BOSE–EINSTEIN ET LASERS À ATOMES
BOSE–EINSTEIN CONDENSATES AND ATOM LASERS

Interactions and entanglements in BECs

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Abstract. In this article we discuss aspects of correlations and entanglements in condensed gases. This requires us to look at the quantum fluctuations in the field that describes the condensates. We discuss ways in which these effects can be observed in experiments and used in precision measurements. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Bose–Einstein condensation / quantum correlations / entanglements / cold collisions

Interactions et intrication dans les condensats de Bose–Einstein

Résumé. *Dans cet article, nous discutons quelques aspects des corrélations et de l'imbrication dans les condensats de Bose–Einstein gazeux. Celà nous amène à nous intéresser aux fluctuations quantiques du champ qui décrit les condensats. Nous discutons des moyens d'observation expérimentalement de tels effets, et de les utiliser dans des mesures de grande précision. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

condensation Bose–Einstein / corrélations quantiques / enchevêtrements / collisions froids

1. Introduction

This article is on the subject of correlations and entanglement between atoms in Bose–Einstein Condensed (BEC) gases. We shall first show how these correlations arise due to the interactions between atoms in a dilute gas. Correlations and entanglements between particles are equivalent to quantum fluctuations in the fields we use to describe condensed gases. Because of this the description of correlations between particles in condensates is linked to that of quantum atomic optics and quantum information processing.

In the first part we shall show how collisions, i.e. binary correlations between atoms, produces the effective interactions between them. To see this we shall consider the short range wavefunction for the relative motion of pairs of atoms at ultra-cold energies. We shall then look at how this binary correlation is modified on length scales comparable with the interparticle distance. This modification is equivalent to configuration interaction in the ground state of the gas. Remarkably, one can also think of it in terms of squeezing of the quantum field of the atoms due to interactions. We shall first show how these correlations

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are included in the formal theory of the dilute Bose gas and how they may be observed in various experiments. These many-body correlations change the effective interactions between the particles and this effect, although small for the three dimensional gases produces a marked effect in 2 dimensions.

In the second part of the article we shall discuss how we can produce useful entanglement through interactions, in particular via four wave mixing of matter waves. This is produced by the same 'Bogoliubov' terms in the Hamiltonian. We shall link this to the generation of entangled pairs by interactions within a BEC.

2. Binary entanglements between ultracold atoms

The theory of condensed gases uses an effective interaction between the ultra-cold atoms. This effective interaction describes the interaction between atoms in the limit of s-wave scattering i.e. when the de Broglie wavelength is much larger that the range of the potential. Atoms cooled by evaporation at sub-micro-kelvin temperatures, have de Broglie wavelengths that are very large compared to the range of the interatomic potential. A local, delta-function, interaction proportional to the scattering length, can be used to describe their interaction. This potential, really a type of pseudo-potential, has the form:

$$V(\boldsymbol{r}-\boldsymbol{r}') = U_0\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1}$$

Here U_0 is given in terms of the binary s-wave scattering length a by $U_0 = 4\pi\hbar^2 a/m$ [1,2]. This effective interaction only describes the coupling between low momentum states and we shall see how it enters mean-field theory of condensates. We shall then look at the corrections to this simplest picture that depends on correlations beyond the binary collision approximation. We can see how the contact interaction describes the increase in the energy of the energy of a condensate by relating it to the scattering wavefunction for a pair of atoms colliding in a condensate. In the first approximation, we assume that the wavefunction has the same form it does in vacuo. For zero collision energy the wavefunction has the simple form which vanishes at a distance equal to the scattering length, i.e.:

$$\chi\left[1-\frac{a}{r}\right] \tag{2}$$

Here a is the scattering length and χ is the asymptotic value of the wavefunction. We now want to show how this form of the wavefunction affects the energy of a gas. It increases the kinetic energy in much the same way an excluded volume inside $r = a$ would do. This extra kinetic energy in the wavefunction is given by:

$$\int_a^\infty \mathrm{d}r\,4\pi r^2\,\frac{\hbar^2}{m}\left\{\chi\nabla\left[1-\frac{a}{r}\right]\right\}^2 = U_0\chi^2 \tag{3}$$

If we take χ^2 as the density of 'other particles' we obtain an expression for the energy of a given particle in the presence of others in the gas. The expression we get is the exactly the same as we would obtain by using first-order perturbation theory with the contact interaction. We should emphasise that this form of the wavefunction at small internuclear distances corresponds to purely binary correlations between the particles. One could also speak of the small distance entanglement between the particles. This is the point of view we want to develop, i.e. entanglements in condensates produced by collisions.

So we now have the form of the effective low energy interaction between particles needed to define the mean-field, Hartree–Fock theory of condensates. For condensates this leads to a non-linear Schrödinger equation called the Gross–Pitaevskii equation, or GPE. The presence of the condensate changes this effective interaction in a way that can be calculated directly for a dilute gas. For three dimensional condensates we should expect this correction to be very small, i.e. of order $n_0a^3 = (a/l_0)^3$, where l_0 is the average interparticle separation. One can see how this arises by taking the upper limit of the integral, which gives the increase in the kinetic energy, to be the healing length in the gas, rather than ∞. This change in the strength of the effective interaction is equivalent to a renormalisation of the interaction strength relative to its value in vacuo. The 'renormalisation' of the effective interaction that we use in the GPE can only be carried out if the gas is dilute and fails for a strongly coupled system such as liquid helium In the next section we shall suppose that the contact interaction is a sensible starting point and examine the simplest mean field theory for condensed gases. We will then return to the formal descriptions of particle entanglement.

3. The Gross–Pitaevskii equation and mean-field theory

The simplest theory of the trapped gases is Hartree–Fock theory where we assume that the atoms all share the same wavefunction i.e. the condensate wavefunction $\Psi(\underline{r},t)$. This is equivalent to neglecting any quantum fluctuations in the gas which come from longer range correlations or entanglement between the atoms. The short range binary correlations are included implicitly through the use of the contact potential. The evolution of the condensate is then described by a non-linear Schrödinger equation called the Gross–Pitaevskii equation (GPE) [1]. In free space this has the form:

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2\nabla^2\Psi}{2m} + U_0|\Psi|^2\Psi \tag{4}$$

This equation has to be solved numerically in general for the case of the trapped gases produced in experiments. This equation has been widely used to predict the properties of trapped condensates with near zero temperatures through the simple addition of the trapping potential to the GP equation, to obtain the following:

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2\nabla^2\Psi}{2m} + V_{\mathrm{trap}}(\boldsymbol{r})\Psi + U_0|\Psi|^2\Psi \tag{5}$$

The agreement between the predictions of this equation for low temperature excitations and the experimental measurements of them is excellent [3]. The corrections to the predictions this equation gives for bulk behaviour such as collective excitations are modest. They are, however, significant for other phenomena as we shall see.

4. Correlations between particles in a condensate many-body effects

When collisions take place in a condensate the other atoms affect the long range wavefunction of the colliding pair. The systematic approach to this problem involves the calculation of the pair correlation in the gas beyond mean-field theory [4–8]. To do this we need to take into account the correction to mean-field theory produced by the quadratic terms in the approximate Hamiltonian i.e. terms of the form:

$$n_0U_0\sum_p\left[a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right] \tag{6}$$

These terms in the Hamiltonian promote pairs of atoms from the condensate into excited states [9]. These pairs deplete the condensate and represent the quantum fluctuations in the mean field from that given by condensate alone. The quadratic terms in the Hamiltonian also change the excited states from particles to quasi-particles. This is represented by the following Bogoliubov transformation:

$$\alpha_p = u_p a_p^\dagger + v_p a_{-p} \quad \text{where} \tag{7}$$

$$u_p = \left[\frac{E_p + \epsilon_p + n_0U_0}{2E_p}\right]^{1/2}, \qquad v_p = -\left[\frac{\epsilon_p + n_0U_0 - E_p}{2E_p}\right]^{1/2} \tag{8}$$

The spectrum of the excitations, i.e. quasi-particles, is given by the well-known Bogoliubov form:

$$E_k = \sqrt{\epsilon_p^2 + 2n_0U_0\epsilon_p} \tag{9}$$

This means that we should consider colliding quasi-particles rather that particles in determining the effective interaction between atoms. This change in the energy spectra of particles is equivalent to changing the 'free' Hamiltonian of the particles, and produces a characteristic change in the wave-function of a pair in the condensate. The effective interaction between particles that includes the change in the pair correlation function can be written in the form:

$$\widetilde{\Gamma}_0 = U_0\left[1 + \widetilde{m}/\psi^2\right] \tag{10}$$

The second term in the effective interaction describes the change in the correlation between atoms because of the Bogoliubov terms in the Hamiltonian. These produce correlations, i.e. pairing, between particles of equal and opposite momenta in the gas. Pairs of this sort become bound in the case of electrons in a superconductor's (or Helium three atoms) below their BCS transition temperature [1]. For the case of a dilute Bose gas they are just an admixture in the ground state of the gas. They are the correlated or configuration interaction part of the ground state wavefunction. $\widetilde{m}$ can be written in the following form:

$$\widetilde{m} = -n_0\widetilde{\Gamma}_0 \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left(\frac{2N(E_k)+1}{2E_k} - \frac{1}{2\epsilon_k}\right) \tag{11}$$

This shows it is a correction to the T-matrix for scattering in a medium [4,5,12]. The fact that we subtract the term with $(1/2\epsilon_k)$ shows that we are calculating the change in the pair correlation from that in vacuo. The change in the pair correlation function corresponds to longer range entanglements between the particles. In the next section we shall look at how we could observe these longer range correlations in the output of an atom laser. Ideally we would like to be able to produce and control the flow of entangled atoms. One of the clearest ways to see these correlations is discussed in the work by Cohen-Tannoudji [10] and Ketterle [11] (this issue).

5. Entangled pairs in condensates and atom laser output

The output of an atom laser [13–15] may be used to probe excitations in the atomic gas, just as quantum evaporation from the surface of ^{4}He can be used to probe its internal dynamics [16–19]. It is also possible, in the case of the dilute Bose gas, to see effects that arise from the correlations in the ground state. For weak output coupling of a trapped atomic gas below the critical temperature T_c, one finds three components that contribute to the output: (i) coherent coupling of the condensate; (ii) stimulated quantum evaporation of thermal excitations; (iii) the simultaneous appearance of an output atom and an internal excitation, indicating the existence of pair entanglements of atoms in the ground state of the gas. The latter is equivalent to the squeezing of the quantum field that describes the atoms.

To see how these processes arise, we consider output coupling from a trapped atomic level $|\mathrm{t}\rangle$ to a free state $|\mathrm{f}\rangle$. Typical mechanisms are a direct (one-photon) radio-frequency transition and an indirect (two-photon) stimulated Raman transition. This coupling can be represented by the following Hamiltonian:

$$H_\text{couple} = \hbar \int \mathrm{d}^3\boldsymbol{r}\,\lambda(\boldsymbol{r},t)\widehat{\psi}_\mathrm{f}^\dagger(\boldsymbol{r})\widehat{\psi}_\mathrm{t}(\boldsymbol{r}) + \text{h.c.} \tag{12}$$

Here $\widehat{\psi}_\mathrm{t}(\boldsymbol{r})$ describes the annihilation of a trapped atom at $\boldsymbol{r}$, while $\widehat{\psi}_\mathrm{f}^\dagger(\boldsymbol{r})$ describes the creation of a free atom with amplitude $\lambda(\boldsymbol{r},t)$. The coupling amplitude can be written as $\lambda(\boldsymbol{r},t) = \tilde{\lambda}(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\mathrm{em}\cdot\boldsymbol{r} - \Delta_\mathrm{em}t)}$ where $\tilde{\lambda}$ is slowly varying in space and time. Here $\hbar\boldsymbol{k}_\mathrm{em}$ and $\hbar\Delta_\mathrm{em}$ are the momentum and energy transferred to an output atom. The full field operator can be written in terms of the annihilation operators $\hat{\alpha}_0$ and $\hat{\alpha}_j$ of the condensate atoms and elementary excitations respectively, obtained using finite temperature form of the Bogoliubov approximation [20,21]. The field operator for the trapped atoms can be written in the form:

$$\widehat{\psi}_{\mathrm{t}}(\boldsymbol{r},t) = \mathrm{e}^{-\mathrm{i}\mu t/\hbar}\left\{ \Psi_0^{\mathrm{t}}(\boldsymbol{r})\hat{\alpha}_0 + \sum_j \left[u_j^{\mathrm{t}}(\boldsymbol{r})\hat{\alpha}_j - v_j^{\mathrm{t}*}(\boldsymbol{r})\hat{\alpha}_j^{\dagger}\right]\right\} \tag{13}$$

Here, μ is the chemical potential, $\Psi_0^{\mathrm{t}}(\boldsymbol{r})$ is the wavefunction of condensate atoms and $u_j^{\mathrm{t}}(\boldsymbol{r}), v_j^{\mathrm{t}}(\boldsymbol{r})$ are found in the absence of output coupling [20,21]. In thermal equilibrium and using the Bogoliubov approximation $\hat{\alpha}_0$ is replaced by $\sqrt{N_0}$, N_0 being the macroscopic mean number of condensate atoms. The population of the excited modes is given by the Bose–Einstein distribution $N_j^{\mathrm{eq}} \equiv \langle \hat{\alpha}_j^{\dagger}\hat{\alpha}_j\rangle_{\mathrm{eq}} = [\mathrm{e}^{\hbar\omega_j/k_{\mathrm{B}}T} - 1]^{-1}$. Equation (13) implies that the annihilation of a trapped particle at $\boldsymbol{r}$ is equivalent to the annihilation of a particle in the condensate (with amplitude $\Psi_0^{\mathrm{t}}(\boldsymbol{r})$), or the annihilation of an elementary excitation (with amplitude u_j^{t}) or the *creation* of a new excitation (with amplitude $-v_j^{\mathrm{t}*}$). The last amplitude is associated with the non-condensate component of the ground-state of the system, which contains entangled pairs of correlated atoms in excited single-particle states, formed by binary collisions. We saw above how this contributes to the anomalous average and changes the pair correlations in the gas. Its role in the output spectrum, as shown below, is the simultaneous creation of an excitation within the trap and along with an output atom. The Bogoliubov functions $u_j^{\mathrm{t}}(\boldsymbol{r})$ of a dilute weakly interacting Bose gas are nearly the same as the energy eigenfunctions $\phi_n(\boldsymbol{r})$ of a single particle in the effective trapping potential $V_{\mathrm{t}}(\boldsymbol{r}) + 2U_0N_0|\Psi_0^{\mathrm{t}}(\boldsymbol{r})|^2$. A formal solution of the Bogoliubov–de Gennes equations yields an expression for the functions $v_j^{\mathrm{t}}(\boldsymbol{r})$ in terms of $u_j^{\mathrm{t}}(\boldsymbol{r})$, i.e. [20,21]:

$$v_j(\boldsymbol{r}) = U_0N_0\sum_n \frac{\int \mathrm{d}^3\boldsymbol{r}'\,\phi_n^*(\boldsymbol{r}')[\Psi_0^*(\boldsymbol{r}')]^2u_j^{\mathrm{t}}(\boldsymbol{r}')}{E_n + \hbar\omega_j}\phi_n(\boldsymbol{r}) \tag{14}$$

This shows that the function $-v_j^*(\boldsymbol{r})$ describes collisional scattering of two atoms in the condensate into excited states producing pairing effects in the ground state of a Bose gas. We described them above in terms of the anomalous average and the change in the pair correlation function for atoms colliding in a condensate. In this part of the article we want to show how they can be observed in the spectrum of the output atoms. After some time t the mean number of output atoms in a state with momentum $\boldsymbol{k}$ is given by $n_{\boldsymbol{k}} = \langle \hat{b}_{\boldsymbol{k}}^{\dagger}\hat{b}_{\boldsymbol{k}}\rangle$, where $\hat{b}_{\boldsymbol{k}} \equiv \int \mathrm{d}^3\boldsymbol{r}\,\varphi_{\boldsymbol{k}}^*(\boldsymbol{r})\widehat{\psi}_{\mathrm{f}}(\boldsymbol{r})$. From the form of the output operator $\widehat{\psi}_{\mathrm{f}}(\boldsymbol{r})$, we obtain:

$$\begin{aligned} n_{\boldsymbol{k}}(t) = {}& \big|\widetilde{\Psi}_{0\boldsymbol{k}}^{\mathrm{f}}\big|^2 N_0^{\mathrm{eq}}\mathcal{D}\big(\omega_{\boldsymbol{k}} - \omega_0^{\mathrm{f}}, \gamma_0/2\big) + \sum_j \Big[\big|\tilde{u}_{j\boldsymbol{k}}^{\mathrm{f}}\big|^2 N_j^{\mathrm{eq}}\mathcal{D}\big(\omega_{\boldsymbol{k}} - \omega_{j+}^{\mathrm{f}}, \gamma_j/2\big) \\ & + \big|\tilde{v}_{j\boldsymbol{k}}^{\mathrm{f}*}\big|^2\big(N_j^{\mathrm{eq}} + 1\big)\mathcal{D}\big(\omega_{\boldsymbol{k}} - \omega_{j-}^{\mathrm{f}}, \gamma_j/2\big)\Big] \end{aligned} \tag{15}$$

Here the time-dependent spectral line shapes

$$\mathcal{D}(\omega, \gamma/2) = \frac{|1 - \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma t/2}|^2}{\omega^2 + \gamma^2/4}$$

tend to Lorentzians of width $\gamma/2$ in the limit $t \gg \gamma^{-1}$.

The first term in equation (15) describes a coherent output component generated when atoms with energy μ in the trapped condensate are excited to a free momentum state with kinetic energy $\hbar\omega_{\boldsymbol{k}} = \mu + \hbar\Delta_{\mathrm{em}}$ by absorbing energy $\hbar\Delta_{\mathrm{em}}$ from the output coupling field. The second term describes a thermal output component generated by stimulated quantum evaporation, where an atom initially populating an elementary excitation with energy $\mu + \hbar\omega_j$ leaves the trap with kinetic energy $\hbar\omega_{\boldsymbol{k}} = \mu + \hbar\omega_j + \hbar\Delta_{\mathrm{em}}$. The third term describes a process where the energy $\hbar\Delta_{\mathrm{em}}$ is sufficient to excite a ground state atom to a free momentum state with kinetic energy $\mu + \hbar\Delta_{\mathrm{em}} - \hbar\omega_j$, leaving a counterpart atom in an excited trap state with energy $\mu + \hbar\omega_j$. While a quantum of elementary excitation is annihilated from the trap in the second process of quantum evaporation, the third term describes a process where a new quantum of elementary excitation (a quasi-particle) is created. The factor $N_j^{\mathrm{eq}} + 1$ in this term implies that this process occurs also in the absence of excitations (at $T = 0$), but it is amplified by the presence of excitations. This process arises directly from the pairs of correlated atoms in the ground state in the trap, which are broken when one atom is forced out. This theory shows that the measurement of the spectrum of a weakly coupled output as a function of the output coupling parameters may be used to study the quantum state of a trapped atomic Bose gas at a finite temperature. The contribution to the output from pair breaking in the ground state shows it contains features that diagnose the quantum fluctuations in the condensate.

6. Entanglement in four wave mixing

So far we have seen how the interactions between atoms in a condensate produce entanglement between them. In the description of output coupling we saw how it may be possible to probe this pairing through the choice of the right parameters for the output coupling field. In the second part of this article we shall discuss the production and characterisation of entangled beams of particles using four-wave mixing of matter waves. The first experiment in the field was carried out at NIST [22]. The principle of the experiments can be explained by analogy with the optical case. The nonlinearity in the matter wave case comes from atomic interactions rather than from the nonlinear response of the medium that the wave is travelling in. The resultant nonlinear Schrödinger equation is, however, very similar to the optical case. In this article we want to discuss the quantum nature of the output beams produced in the four-wave mixing process. We have already seen the sort of terms in the Hamiltonian that are important for generating the entangled beams. This is the Bogoliubov terms in the Hamiltonian, which for two modes has the form:

$$\kappa a^{+} b^{+} + \kappa^{*} a b \tag{16}$$

In the case of the quadratic terms that promoted pairs from the condensate the coupling is not resonant and we just find an admixture of pairs of atoms into the ground state. This is the term in the Hamiltonian that we had to include in our description of the gas to go beyond the simplest version of mean-field. It was also responsible for the Beliaev damping process i.e. down-conversion of a single quasi-particle into a pair: the only process that can damp an excitation at zero temperature.

To see resonant coupling from condensates we need to start with colliding coherent beams, where energy and momentum are available to produce real pairs. In that case we can think of the quadratic Hamiltonian as representing the transfer of two particles in the colliding beams into two other states labelled by a and b. This process will produce appreciable transfer of pairs to the a and b beams when we have energy and momentum conservation holding for the process. This is why it only produced an admixture of excited states into the ground state of the system in the case of a static condensate.

In the case of colliding condensates one produces entangled pairs of atoms scattered in all directions. One can stimulate the process through the use of a third beam in a four wave mixing process. This produces beams where a portion of the atoms is entangled [23]. The maximum entanglement that can be achieved in experiments of this sort is the subject of current investigations. The modes are transformed under the influence of the Hamiltonian into the well known [24] Bogoliubov form:

$$a(t) = a\cosh(r) - b^{\dagger}\exp(\mathrm{i}\phi)\sinh(r), \quad \text{where} \tag{17}$$

$$r \sim n_0 U_0 t/\hbar \tag{18}$$

The phase ϕ depends on the phase of the pump modes i.e. the colliding condensates, relative to those in which the pairs are grown. The state of the field produced by this Hamiltonian has the form:

$$|\zeta_{AB}\rangle = \mathrm{sech}(r) \sum_{n=0}^{\infty} \big[-\exp(\mathrm{i}\phi)\tanh(r)\big]^{n} |n, n\rangle \tag{19}$$

This field is squeezed in the sense that the difference in the number of bosons in each mode is reduced relative to the independent beam limit, where the difference distribution would have a width roughly equal to the square root of the mean number of atoms in each beam. The squeezing in the relative number makes the twin beam useful in an interferometer. This is also termed spin squeezing for some applications, e.g. in the discussion of clocks. In fact the states produced have zero variance in the number difference and are, therefore optimally spin squeezed. The square of the variance in $(n_a - n_b)$ can be written in the form:

$$\Delta(n_a - n_b)^2 = \tfrac{1}{4}\left[\left\langle\left(a^\dagger a - b^\dagger b\right)^2\right\rangle - \left\langle a^\dagger a - b^\dagger b\right\rangle^2\right] \tag{20}$$

In the case of the state produced by spontaneous four wave mixing the mean value is zero and we have the form:

$$\Delta(n_a - n_b)^2 = \tfrac{1}{4}\left\langle\left(a^\dagger a - b^\dagger b\right)\left(a^\dagger a - b^\dagger b\right)\right\rangle = \tfrac{1}{4}\left[\Delta n_a^2 + \Delta n_b^2 - 2\left(\langle n_a n_b\rangle - \langle n_a\rangle\langle n_b\rangle\right)\right] \tag{21}$$

For independent thermal states in the two modes this would be:

$$\left[n_a(n_a+1) + n_b(n_b+1)\right]/2 \tag{22}$$

The correlations between the modes, represented by the maximal value of $\langle n_a n_b\rangle - \langle n_a\rangle\langle n_b\rangle$ is what produces the reduced value of the fluctuations. We can relate this to the pairing fields for the modes as follows. We define the pairing field by:

$$m = \langle ab\rangle \tag{23}$$

When it has a non-zero values it measures the number of pairs in the two modes. For the field produced by spontaneous four wave mixing we can see its role by noting that:

$$mm^* = \left(\Delta n_a^2 + \Delta n_b^2\right) \sim 2n^2 \tag{24}$$

since,

$$\Delta(n_a - n_b)^2 = 0 \tag{25}$$

This squeezing is important for interferometric measurements based on boson beams. We make it useful after generation, by putting the two beams onto the ports of a beam splitter. The spread in number of atoms between the resultant beams, within the interferometer, is the order of the number of the atoms present. We can then infer the phase sensitivity of the interferometer using:

$$\Delta n \Delta\phi \sim 1 \tag{26}$$

For the maximally squeezed state we have in the arms of the interferometer:

$$\Delta n \sim n \tag{27}$$

and we thus have:

$$\Delta\phi \sim 1/n \tag{28}$$

This is the Heisenberg limit for phase measurements and can only be reached in perfect conditions.

7. Effects of dissipation and imperfect detection

In principle, states with squeezed relative number fluctuations allow one to make measurements with interferometers below the standard quantum limit. The achievement of enhanced operation of interferometers depends, however, on the effects of decay as well as the efficiency with which we can detect particles. If we start with precisely matched numbers in each beam as described above, we can achieve the Heisenberg limit for relative phase measurements. This situation is rapidly changed by the presence of particle loss. We lose the Heisenberg limit on a timescale that is the inverse of N times the individual collision rate: a very small interval for a large number of atoms. Part of this loss is due to collisions and part is due to the fact that we implicitly lose particles if we cannot detect them perfectly. The dominant collisions will be between the emerging entangled beams and the original, and more dense, pump beams. We can then show that the Heisenberg limit is lost in a time of the order the inverse of $N\gamma_c$, showing that the Heisenberg limit will be very difficult to achieve in a practical device. The advantage of having partial squeezing of the relative number fluctuations below $\sqrt{N}$, i.e. below that given for independent beams, is a matter of intense investigation at the present time. The relative number squeezed state will evolve on a timescale γ_c^{-1} to a coherent state with no relative squeezing at all. This is the state you have when you start with pre-existing uncorrelated atoms, i.e. coherent states in each of the beams. If you use an injected third wave to stimulate the production of a fourth one you are in effect starting with uncorrelated atoms which you hope to swamp with correlated pairs. The precise limits to which squeezing can be achieved by four-wave mixing in the presence of loss is also a matter of great interest at the present time.

One cannot detect atoms with unit efficiency, and this has been shown to have a profound effect on the enhanced operation with entangled states. In fact, to achieve the Heisenberg limit with n particles we have to have a quantum efficiency around $(1 - 1/n)$. Real efficiencies are typically 0.8. This sets a stringent limit on what we should expect to do in practice.

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