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# Using quantum theory to improve measurement precision

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Progress in science is inextricably linked with how well we can observe the world around us. It is by making increasingly better measurements that scientific theories are tested and refined. In this article we address the question of what the ultimate limit to measurement precision is and how it could be achieved in the laboratory. We focus on how, by making use of quantum theory, it is possible to make better measurements than anything that can be achieved with classical techniques. This opens the door to an array of new technologies and could help answer some of science's most engaging questions.

## 1. Introduction

Measurements are an inherent part of our everyday lives. Every sight, smell, or sound we experience is a measurement of our surroundings. We concern ourselves with such things as how far our commute to work is, how long it will take to find a parking space, whether it is warm enough to take a walk after lunch and, if we do not, how much our waistline is likely to expand.

Measurements have played a key role in the development of business and commerce. They allow traders in different cities to communicate effectively and understand exactly what it is that they have agreed to buy and sell. Reference standards enable comparisons to be made and a lot of work has been done to define units that everyone can agree upon in terms of measurable physical quantities. These new definitions not only improve precision but also allow measuring devices to be calibrated by people the world over and not just by those lucky enough to have a key to the 'world reference vault'.

Measurements are also crucial to science, not just for the important role science plays in defining standards, but in a much more fundamental sense. Indeed it could be said that all of science is just the business of predicting the outcomes of measurements. Or, to take Karl Popper's view, measurements allow us to prove scientific theories wrong and so tighten the net around viable correct theories. Science should not be concerned with what cannot be expressed as a click in a photodetector, the position of a needle in a voltmeter, or the outcome of any other measurement.

In the late nineteenth century, confidence in the success of physics was sky-high and there was a widespread feeling that there was nothing new to be discovered—all that remained was more and more precise measurement. With hindsight it is easy to see that this confidence was misplaced. In fact, it was to be these very measurements that were to reveal new and more fundamental theories of physics. The course of physics is littered with theories that have failed to stand up to the rigours of precise measurements: Ptolemy's geocentric model of the Universe was challenged by Copernicus' model which put the sun at the centre; measurements by Tycho Brahe confirmed the heliocentric model but highlighted further anomalies that set the foundations for Kepler's improved theories of planetary motion; and the long-debated concept of an 'ether' that supported the propagation of light was roundly defeated when Michelson and Morley made precise interferometric measurements. The key to overturning these, and many other, theories has been the careful gathering of precise measurement data.

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In April 1900, Lord Kelvin gave a lecture to the Royal Institution of Great Britain in which he said that the 'beauty and clearness' of physical theories were overshadowed by 'two clouds'. He was talking about the null result of the Michelson – Morley experiment and the problems of black-body radiation. In fact, these 'two clouds' were to herald the early twentieth century revolution in theoretical physics with the emergence of relativity and quantum theory. Today many open questions remain in physics. The public imagination has been captivated by fundamental physical questions such as whether the Higgs Boson exists, how much dark matter there is in the Universe, and whether there are gravitational waves. As has happened time and again through history, there is every reason to believe that these issues will be resolved just as soon as measurement technologies catch up.

A large part of physics, therefore, boils down to finding ways to improve measurements. These allow us to make theories jump through increasingly smaller hoops until eventually cracks are revealed that need to be patched up with new theories. In this paper, I want to turn this traditional route on its head: instead of using measurements to find theories, I want to investigate how new theories can be exploited to improve measurements. Of course, the hope is that this may, in turn, lead to even newer theories and hence even newer measurement techniques. In particular, I want to concentrate on how the advent of quantum theory has opened the door to a whole new raft of precision measurements.

## 2. Limits to measurements

Galileo expressed the aim of experimental science as being to, 'Measure what is measurable, and make measurable what is not so'. While certainly admirable, this sentiment raises the intriguing question of what would happen if there existed some fundamental limit to what we could measure. Such an impasse would threaten to halt the scientific process since it would limit how deeply we could distinguish competing theories.

As a simple example, we can imagine looking at a patterned card through a filter that admits only red light. Suppose that the pattern we see is a 'chequer board' of alternating red and black squares as shown in figure 1. Now suppose that we would like to know the 'true' underlying pattern. One possibility is that it is exactly as observed, i.e. red and black squares. However, there are also other possibilities that give the same observed pattern—two of which are shown in figure 1. If we were not able to remove the filter, these different possibilities would be indistinguishable and a level of structure in the pattern would remain hidden from our view.

Of course, in this example, we can find ways to remove the filter and 'make measurable what is not so'. We might be tempted to say that this is always true: if we are clever enough, dedicated enough, and spend enough money we should be able to make measurements to whatever level of accuracy we want. This certainly was the view at the end of the nineteenth century when Michelson famously remarked that the future of physics lay in looking in 'the sixth place of decimals'. The quantum revolution, however, sharply changed all that.

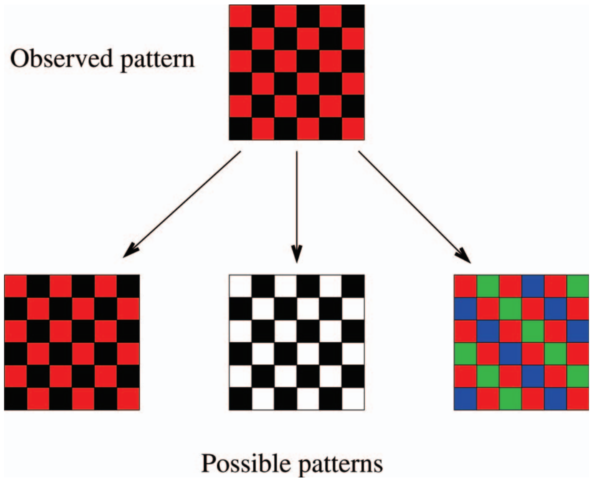


Figure 1. The observed pattern of a card viewed through a red filter is a 'chequer board' of red and black squares. If the filter is removed, the pattern on the card could be one of a number of possibilities, three of which are shown here.

In quantum physics the act of observation changes the system being studied—a feature well understood by the social science community. A classic catch-22 prevents us from knowing what happens in quantum systems when we are not observing them since the only way to find out would be to make a measurement, but the measurement inevitably changes the system. What happens to a photon before it is measured in Young's double slit experiment, for example, is a mystery that remains tantalizingly beyond our grasp. It is part of the unspeakable in quantum physics.

We also know that in quantum physics it is not possible to simultaneously measure two conjugate variables (e.g. position and momentum) with arbitrary accuracy. If we know the position of a particle precisely, then its momentum will be completely unknown and vice versa. This concept is expressed in Heisenberg's famous uncertainty relationship. But what if we are happy to measure just one conjugate variable and do not care about the other one? Surely, then, we can measure this quantity arbitrarily accurately. Unfortunately, even this is not true. We need to think about how the measurement is made. One approach to measuring the position of a particle is to scatter light from it. This means that the position measurement will be limited by such things as diffractive optics that depend on the wavelength of the light used†. Of course we could improve things by using light of progressively shorter wavelengths—going from visible light, to ultraviolet, to X-rays, and beyond. However, to achieve infinite precision, we would need photons with infinite energy. The progression in measurement precision would come to a grinding halt as soon as the photon needed to have more than all the energy in the Universe.

It turns out that a fundamental limit is reached before then. At some point, the level of precision will demand scattered light that is so energetic that it can spontaneously create a new particle out of the vacuum of the type being measured. In such a case, the position of the original particle is ambiguous because now we have two of them. This problem arises when we try to measure the position of a particle to within its Compton wavelength. Equating the Compton wavelength with the Schwarzchild radius of the created particle gives the so-called Planck length (approximately $1.6 \times 10^{-35}$ m). If we try to measure to within a Planck length, the scattered photon creates a black hole and is 'swallowed up' and we are not able to make any measurement. Not surprisingly, the Planck length is a scale beyond which it is widely regarded that traditional notions of space and time break down and it is not meaningful to ask physical questions.

This discussion highlights the fact that measurements are linked with physical theories in an intrinsic way and are not just bolted-on embellishments that allow us to describe experiments. Questions regarding what can and cannot be measured in science have profound significance: they define what lies inside or outside the realm of scientific enquiry. Measurement is a physical process and the need for a self-consistent theory of physics that includes measurement is a fact well known by quantum theorists.

Putting aside for now the question of the fundamental limit to measurement, a more practical (and no less intriguing) question is, 'How does the precision that can be achieved scale with the resources used?'. Or, to put it another way, 'Given fixed resources (e.g. energy), what is the best possible measurement that can be made?'. It is this question that we will turn to now. In particular, we will consider how quantum theory allows us to do better than anything that can be achieved by classical physics. A good place to start this discussion is with atomic fountain clocks—the basis of present primary time standards.

†Interestingly, it has been shown that it may be possible (at least in principle) to perfectly resolve images with light of a given wavelength by making use of lenses constructed from materials with negative refractive indices [1,2]. There will, however, still be limits to how accurately the position of an object can be measured. For example, this approach relies on having lenses with perfect surfaces, but we know that there will be some unavoidable 'jiggle' of the atoms in the lens due to the uncertainty principle.

### 3. Fountain clocks

The best clocks that are currently available are atomic fountains [3–5] based on Ramsey interferometry with separated oscillating fields [6]. These enable the second to be measured with an accuracy better than 1 part in $10^{15}$. They work by comparing the frequency of an oscillator to the transition frequency between the two hyperfine levels of the ground state of $^{133}Cs$. In 1967, the second was defined to be the duration of 9192 631 770 such oscillations. The definition of the second has changed a number of times through history and it is likely that it will be redefined in the future as improved techniques are developed. Recent experiments [7] with single mercury ions, for example, have achieved an order of magnitude improvement over atomic fountains and hold great promise for future time standards.

A fountain clock works by first trapping and cooling a cloud of caesium atoms to micro Kelvin temperatures by an arrangement of three pairs of counterpropagating laser beams in orthogonal directions. By cleverly detuning the laser beams below an atomic transition frequency, the atoms are more likely to absorb photons from beams that oppose their motion since the frequency of these beams are Doppler shifted into resonance. This means that the atoms always feel a drag force that opposes their motion—an effect evocatively termed 'optical molasses'. The cooled atoms are then launched upwards and prepared in one of the hyperfine levels of their electronic ground state. The atoms pass through a microwave cavity (see figure 2) containing a field tuned near to the transition frequency between the ground state hyperfine levels for caesium. They continue moving upwards until the Earth's gravity causes them to fall back through the same microwave cavity. The two exposures to the microwave field cause some atoms to make the transition between the hyperfine states. The frequency of the microwave field relative to the hyperfine frequency can then be inferred by measuring the fraction of atoms in each state after this second transit of the cavity. This allows us to compare the microwave oscillator with the 'universal time standard'.

A beautiful feature of Ramsey's work was that he showed that the uncertainty in the measured frequency scales inversely with the time of flight of the atoms [6]. This means that to increase the measurement precision we need only increase the time of flight by launching the atoms up through the cavity a little faster. Of course, eventually we come up against fundamental, not to mention practical, reasons why this process cannot continue indefinitely.

Two factors which limit the time of flight are the temperature of the atoms and the effects of gravity. The microwave cavity typically has a hole no larger than about 1 cm to ensure it is of high enough quality. The atoms must pass back through it on their return trip, which means they

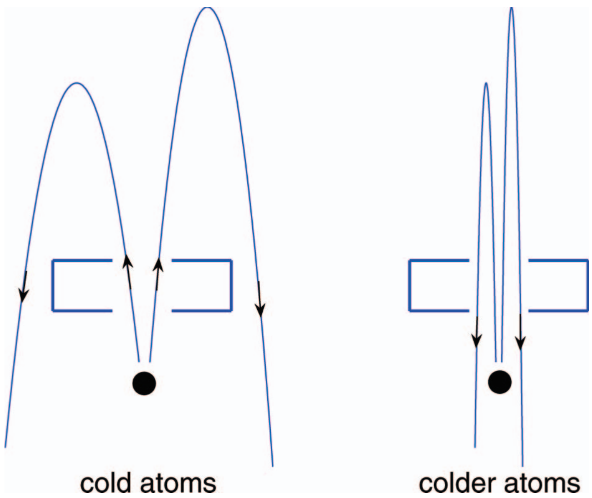


Figure 2. Schematic of an atomic fountain. Laser-cooled $^{133}Cs$ atoms are launched upwards through a cavity containing a microwave field. After some time, gravity reverses their direction and they fall back towards the cavity. If the atoms are not cold enough a significant fraction will miss the hole in the cavity on the return trip. Colder atoms allow a longer time of flight and therefore better measurement resolution.

should not spread more than 1 cm. This restricts the time of flight for a given temperature. For colder temperatures the atoms are more sluggish and spread out more slowly. This allows longer times of flight and therefore better measurement precision (see figure 2). The other limitation is the effect of gravity and the size that atomic fountains can be built in the laboratory. For current fountains, which are about one metre high, the return time is about one second; for significantly longer times, much taller towers would be needed, leading to severe problems with, among other things, magnetic field control and temperature homogeneity [8].

It turns out that on Earth, the effect of gravity is the limiting factor and cooling the atoms using optical molasses is sufficient. We can see this as follows. For a one second return time, we require atoms with a spread of speeds less than 1 cm $s^{-1}$ so that a substantial fraction pass back through the cavity. The mean speed of atoms at a temperature, $T$, is given by $\bar{v} = (3k_B T/m)^{1/2}$, where $m$ is the mass of the atom and $k_B$ is Boltzmann's constant. Putting in numbers for laser-cooled caesium atoms at a temperature of 1 $\mu$K, gives a mean speed of about 1 cm $s^{-1}$ and so we see there is little advantage to cooling them further. A clock in space, by contrast, is not constrained by the effect of gravity and can have a much longer interaction time than is possible on Earth [9,10]. In this case, the colder the atoms the better. This is where Bose–Einstein condensates (BECs) might be useful.

A BEC [11] is a new state of matter that was predicted by Einstein in 1925 [12] and first observed in dilute gases in 1995 [13]. It consists of a large collection of bosons that, if they are cold enough and dense enough, undergo a phase transition to all occupy the ground state of the system. BECs are fascinating because they are an example of a large scale quantum object. We normally associate quantum effects only with microscopic objects. However, BECs exhibit quantum behaviour and yet consist of a large number of particles (up to billions) and have a large spatial extent of typically 100 $\mu$m. That's about the thickness of a human hair and certainly of a size that could be seen with the naked eye. Perhaps the most interesting property of BECs for use in clocks is the fact that they are extraordinarily cold—typically a few billionths of a degree above absolute zero—and so expand very slowly. In principle, this should allow observation times longer than 1000 s and a significant improvement in the accuracy of clocks. The rate at which a BEC expands is given by Heisenberg's uncertainty principle. The more tightly that a BEC is initially confined in a particular direction, the larger the spread of momenta in that direction. This means that a BEC spreads more rapidly in directions that were initially tightly confined than in those that were not. This is in contrast to a thermal cloud of atoms which expands isotropically regardless of the shape of the initial trapping potential and is a beautiful demonstration of the uncertainty principle in action. Conveniently, BECs have already been created with $^{133}Cs$ atoms [14], the very element that defines the SI second, and so they hold great promise for further improving time standards. BECs are truly quantum objects and provide just one example of how quantum physics may be used to improve measurement precision.

So far, our discussion of atomic clocks has focused largely on making the time of flight as long as possible while ensuring that as many atoms as possible pass back through the hole in the cavity. But why should it matter if we miss some? Obviously, we do not want to lose all the atoms or we could not make any measurement, but we might expect that so long as a few make it we are still in business. After all, the atoms are independent of one another and so each must contain all the information we want. In principle this argument is correct but a problem arises when we go to measure the atoms. Whenever we detect the state of a caesium atom, we find it in either of the two relevant hyperfine levels. The fact that we never find atoms in states between these two extremes introduces some additional noise into the measurement. We can understand this with a simple analogy.

Suppose we are given a coin and want to find out whether or not it is 'fair', i.e. it has a 50:50 chance of coming up heads or tails. One way is to try tossing it. If on the first toss it comes up (say) heads, we are no nearer to a conclusion. We already knew it had to be one or the other and we have no idea whether each of the next hundred tosses will also be heads or whether there will be an even distribution of heads and tails. As more tosses are made, we get a better idea of the probability of each outcome since the fractional width of the distribution decreases. The more tosses that are made (or caesium atoms that are detected) the better the precision of the measurement. This effect is an artifact of the two outcomes being quantized. If on a single toss of the coin we could measure it as (say) 83% heads and 17% tails then we would immediately know the two probabilities. Of course, in reality, we know that a single toss always comes up completely heads or completely tails. This partitioning of the outcomes is the source of the additional noise. This so-called 'shot noise' also occurs in quantum systems and a good way to understand it is to consider a Mach–Zehnder interferometer.

## 4. Interferometers and shot noise

The interferometer has been one of the most significant developments in metrology. It has enabled path length differences to be detected with unprecedented accuracy and has played a key role in a number of exciting experiments. Interferometers enabled Michelson and Morley to disprove the existence of the ether and giant versions of the same set-up, with four kilometre long arms, are now being used to look for evidence of gravitational waves at LIGO in Washington and Louisiana [15]. If successful, these experiments would confirm predictions made by Einstein's general theory of relativity and be another in a long list of triumphs for interferometry. Despite their great successes, however, interferometers are limited by the same source of noise that affects coin-tossing measurements and time-keeping with atomic clocks and it is worthwhile studying how this noise comes about.

A Mach–Zehnder interferometer consists of two 50:50 beam splitters and a phase shift, $\phi$, on one of the paths (see figure 3). This phase shift is equivalent to a path length

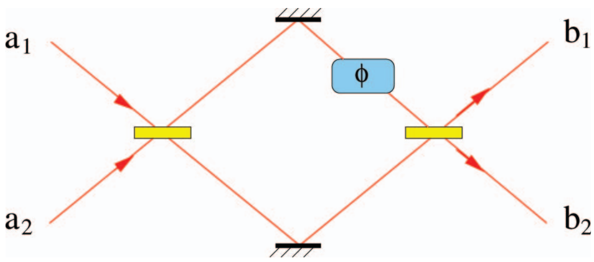


Figure 3. A Mach-Zehnder interferometer consisting of two mirrors, two 50:50 beam splitters, and a phase shift, $\phi$ on one arm. If a photon enters at port $a_1$, it will be detected in ports $b_1$ and $b_2$ with probabilities $\sin^2(\phi/2)$ and $\cos^2(\phi/2)$ respectively.

difference between the arms. To make things simple, we will consider mirrors that are sufficiently massive that we can neglect the effects of fluctuations in radiation pressure of the light passing through. In some schemes these effects can be important and need to be taken into account [16–18].

The overall effect of the interferometer is to transform the annihilation operators at the input ports, $a_1$ and $a_2$, into annihilation operators at the output ports, $b_1$ and $b_2$, by the following transformation,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \sin(\phi/2) & \cos(\phi/2) \\ \cos(\phi/2) & -\sin(\phi/2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (1)$$

If a particle enters the interferometer at port $a_1$, i.e. the input state is $|1,0\rangle = a_1^\dagger|0,0\rangle$, the output state is found by transforming the operator $a_1$ using (1) to give $|\psi\rangle = \sin(\phi/2)|1,0\rangle + \cos(\phi/2)|0,1\rangle$. The probabilities that the particle emerges in ports $b_1$ and $b_2$ are then, $P_{b_1} = \sin^2(\phi/2)$ and $P_{b_2} = \cos^2(\phi/2)$.

The phase shift can be found by measuring the difference, $n$, in the number of particles detected in the two output ports,

$$\begin{aligned} \langle n\rangle &= \langle b_1^\dagger b_1\rangle - \langle b_2^\dagger b_2\rangle \\ &= \left(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle\right)\cos\phi + \left(\langle a_1^\dagger a_2\rangle + \langle a_2^\dagger a_1\rangle\right)\sin\phi. \end{aligned} \quad (2)$$

Throughout this discussion, I will consider only input states for which $\langle a_1^\dagger a_2\rangle = \langle a_2^\dagger a_1\rangle = 0$, which means we can write

$$\langle n\rangle = \left(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle\right)\cos\phi. \quad (3)$$

Normally in interferometry, a coherent state [19], $|\alpha\rangle$, is fed into one port, say $a_1$, and a vacuum state, $|0\rangle$, is fed into the other. A measurement at the output ports then gives $\langle n\rangle = N\cos\phi$, where $N = |\alpha|^2$ is the mean number of particles in the coherent state. This illustrates the principle behind an interferometer: by measuring the difference in the number of particles at each output port we can infer the phase shift to within $2\pi$, i.e. the path length difference can be measured to within one wavelength of the light used. As discussed above, one possible way of improving the measurement precision is to use light with a shorter wavelength. Here we would like to consider how accurately we can measure a phase for a given wavelength.

The variance of $n$ is given by

$$(\Delta n)^2 = \langle(a_1^\dagger a_2 + a_2^\dagger a_1)^2\rangle \sin^2\phi, \quad (4)$$

and is related to the uncertainty in our measurement of $\phi$ by a simple manipulation of the errors,

$$(\Delta n)^2 = \left(\frac{\partial\langle n\rangle}{\partial\phi}\right)^2 (\Delta\phi)^2. \tag{5}$$

Rearranging and substituting for $\langle n\rangle$ and $(\Delta n)^2$, from equations (3) and (4), we obtain an expression for the uncertainty in $\phi$,

$$(\Delta\phi)^2 = \frac{\langle (a_1^\dagger a_2 + a_2^\dagger a_1)^2\rangle}{(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle)^2}. \tag{6}$$

It is notable that the measurement precision $\Delta\phi$ is independent of the value of $\phi$. If we now consider the case of standard interferometry where a coherent state, $|\alpha\rangle$, is fed into one input port and a vacuum state, $|0\rangle$, into the other, we get $(\Delta\phi)^2 = 1/|\alpha|^2 = 1/N$.

This means that an interferometer can measure a phase shift with a precision that scales as $1/N^{1/2}$. This is the well-known scaling of shot noise with particle number. The measurement precision improves as more particles are detected and so it is possible to improve our measurement of $\phi$ simply by using more particles. This explains why we want as many atoms as possible to return through the microwave cavity in a fountain clock set-up and why we want as many tosses of a coin as is practical to determine its fairness. Shot noise can be a major limitation to precision measurement schemes and we would now like to consider ways this problem can be overcome.

## 5. Squeezed states

Up until now we have considered only coherent states as the input to the interferometer. At one port we had a coherent state with amplitude $\alpha$ and at the other we had a vacuum state, which is just a coherent state with zero amplitude. In a sense, these can be thought of as the 'most classical' states of light. It is therefore of interest to ask whether using non-classical states of light can improve the sensitivity of interferometers.

A convenient way to describe states of light is to use the quadrature operators defined by $X_1 = (a + a^\dagger)$ and $X_2 = -\mathrm{i}(a - a^\dagger)$ [20]. These satisfy the commutation relation $[X_1, X_2] = 2\mathrm{i}$ and their variances satisfy the uncertainty relation $\Delta X_1 \Delta X_2 \geq 1$. Coherent states are minimum uncertainty states, i.e. $\Delta X_1 \Delta X_2 = 1$, and have the additional feature that the uncertainties of the quadrature operators are equal,

$$\Delta X_1 = \Delta X_2 = 1. \tag{7}$$

In figure 4(*a*) the mean amplitudes and associated uncertainties for a coherent state and a vacuum state are represented as error circles. The centre of each circle is the amplitude of the corresponding state and the diameter of the circle represents the quadrature-operator uncertainties. To simplify things, the phase of the coherent state is taken to be zero.

Of course, the uncertainties in each quadrature do not have to be equal. It is possible to have a minimum uncertainty state with smaller fluctuations in one quadrature than a coherent state. This reduction in one quadrature

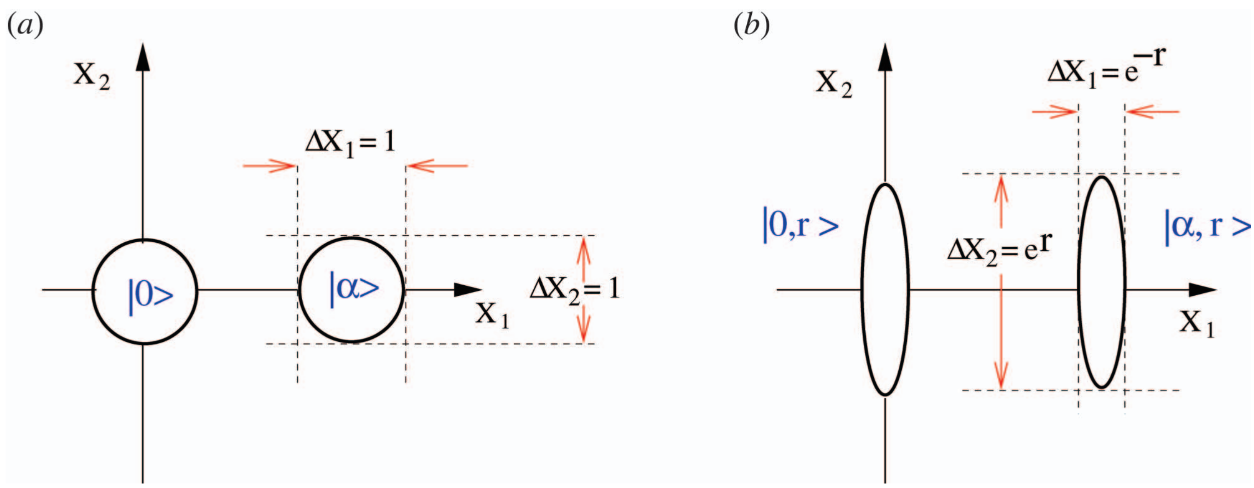


Figure 4. Phase space representation of the mean values and uncertainties of $X_1$ and $X_2$ for (*a*) a vacuum state, $|0\rangle$, and a coherent state, $|\alpha\rangle$, and (*b*) a squeezed vacuum, $|0, r\rangle$ and a squeezed coherent state, $|\alpha, r\rangle$, where $r > 0$ is the squeeze parameter.

must be accompanied by an equivalent increase in the fluctuations in the other quadrature to ensure that the uncertainty relation is not violated. Such a state is known as a squeezed state [21,22]. Squeezed states may be defined by two quantities: their amplitude, $\alpha$, and their squeeze parameter, $\zeta = r\ \exp(i\theta)$, where $r$ is the strength of the squeezing and $\theta$ gives the direction of the squeezing. For simplicity, throughout this discussion we will take $\theta = 0$, i.e. the state is squeezed in the $X_1$ quadrature and we can take $\zeta = r$. Phase space representations of a squeezed vacuum, $|0, r\rangle$, and a squeezed state with amplitude $\alpha$, $|\alpha, r\rangle$ are depicted in figure 4(*b*). The uncertainties in the two quadratures for states with a squeeze parameter $r$ are $\Delta X_1 = \exp(-r)$ and $\Delta X_2 = \exp(r)$. It is clear that these are minimum uncertainty states since $\Delta X_1 \Delta X_2 = 1$.

The first experimental realization of squeezed light was reported by Slusher and co-workers in 1985 [23]. They were able to demonstrate a 7% to 10% reduction in the noise of one quadrature below the level of a coherent state. Since then, squeezing has developed into a mature field and has been proposed and experimentally demonstrated in many different optical and atomic schemes. The degree that states can be squeezed has also progressed in leaps and bounds and it is now even possible to create 'perfectly squeezed' number states that have essentially no uncertainty in the number quadrature [24].

One of the most exciting possibilities for squeezed states is to use them for precision measurements. The fact that the noise in one quadrature can be much less than a coherent state suggests that they might allow us to beat the shot-noise limit. One 'obvious' thing to try is to use a squeezed state $|\alpha, r\rangle$ instead of a coherent state, $|\alpha\rangle$, as the input to a Mach–Zehnder interferometer. Let us now consider how this works.

Suppose we feed the state $|\alpha, r\rangle$ into port $a_1$ and the vacuum $|0\rangle$ into port $a_2$. The precision with which $\phi$ can be measured is given by equation (6). Rearranging the order of the operators a little, we get

$$(\Delta\phi)^2 = \frac{\langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle\left(2\langle a_1^\dagger a_1\rangle + 1\right)}{(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle)^2} = \frac{1}{\langle a_1^\dagger a_1\rangle}, \qquad (8)$$

where the last equality follows since $\langle a_2^\dagger a_2\rangle = 0$, i.e. there are no particles in the vacuum. This means that the phase resolution is given by $\Delta\phi = 1/N^{1/2}$ and, despite all our efforts, we are back to the shot-noise limit and no better off than if we had used an unsqueezed coherent state all along.

All is not lost however. A key breakthrough was made when Caves realized that, somewhat counterintuitively, we should squeeze the unused (vacuum) port rather than the coherent state input [16,18]. Suppose we feed the coherent state $|\alpha\rangle$ into port $a_1$ and the squeezed vacuum $|0, r\rangle$ into port $a_2$. Rearranging (6), the measurement resolution is given by

$$(\Delta\phi)^2 = \frac{N(\Delta X_1)^2 + \langle a_2^\dagger a_2\rangle}{(N - \langle a_2^\dagger a_2\rangle)^2}, \qquad (9)$$

where $N = |\alpha|^2$. We can substitute $(\Delta X_1)^2 = \exp(-2r)$ and replace $\langle a_2^\dagger a_2\rangle$ with the mean number of photons in a squeezed vacuum. It turns out that this is no longer zero, as it is for an unsqueezed vacuum, but rather $\langle a_2^\dagger a_2\rangle = \sinh^2 r$. This gives

$$(\Delta\phi)^2 = \frac{N\exp(-2r) + \sinh^2 r}{(N - \sinh^2 r)^2}. \qquad (10)$$

For modest levels of squeezing, $N \gg \sinh^2 r$, we get $\Delta\phi \approx \exp(-r)/N^{1/2}$. This is smaller than the shot-noise limit just as we want and this improved resolution has been observed in experiments [25,26].

The resolution cannot be improved indefinitely by squeezing the vacuum harder and harder. At some point a balance will be reached between reducing the vacuum fluctuations, which tends to improve the resolution, and increasing the mean number of photons in the vacuum port, which tends to degrade the resolution. Minimizing $\Delta\phi$ as a function of $r$ gives the limit to the phase resolution that can be attained by this method. For large numbers of particles, $N \gg 1$, this is

$$\Delta\phi \approx N^{-3/4}, \qquad (11)$$

which is clearly an improvement on the shot-noise limit and demonstrates that quantum theory can be used to improve measurement precision. This, however, is not the ultimate limit and it should be possible to do even better.

## 6. The cat gets the cream

Particle number and phase obey a kind of uncertainty relation, $\Delta N \Delta\phi \gtrsim 1$, in the sense that as the number uncertainty is increased, the phase uncertainty goes down and vice versa. This apparent relationship was given a firm footing by Summy and Pegg [27] and is interesting from the viewpoint of precision measurements since it tells us how accurately a phase can be measured with a given resource of particles. Suppose we had a total of $N$ particles available. The maximum number uncertainty that any state in this system can have is $\Delta N = N$, since it can have anything from zero up to $N$ particles. The uncertainty relation suggests that the minimum resolvable phase is then $\Delta\phi \sim 1/N$, which is better than both the shot-noise and the squeezed light limits discussed above. This is commonly referred to as the 'Heisenberg limit' due to the Heisenberg-like uncertainty relation on which it is based.

So how do we create a state with the maximum possible number uncertainty? One possibility is to create a 'macroscopic superposition'. These states are also called Schrödinger cat states in analogy with Schrödinger's famous thought experiment in which he showed that a bizarre consequence of quantum theory was that it should be possible to put a cat into a macroscopic superposition of being alive and dead [28]. We can see how such macroscopic superpositions allow us to maximize the number uncertainty as follows. Suppose, for argument's sake, we wanted to measure a phase and had at our disposal 60 carbon atoms. One strategy would be to send the atoms through an interferometer one by one and measure the number that emerges from each output port. This would allow us to make a shot-noise limited measurement as discussed above. Another possibility would be to 'stick' all the atoms together to make a $C_{60}$ buckyball molecule and then to send this molecule through the interferometer. Inside the interferometer there would be a Schrödinger cat-like superposition of all the atoms on one path and all on the other. The uncertainty in the number of particles on each path is now $N = 60$ (since we cannot know in advance whether the outcome of a measurement on either path will be zero or 60) and the phase uncertainty is $\Delta\phi \sim 1/N$ just as we want. The difference in the mean number of molecules detected at each output port, in this case, can be shown to be

$$\langle n \rangle = \cos(N\phi). \tag{12}$$

Comparing this with the result for individual particles (3), we see that the phase is amplified by a factor of $N$, which means that we should be able to measure phase shifts that are $N$ times smaller. One way to understand this enhancement is that, if the molecule travels along one path, every one of the constituent atoms will get a phase shift, whereas if it travels along the other path, none of them do. This is why the phase shift for a molecule is $N$ times bigger than for an individual particle. An alternative explanation is in terms of the de Broglie wavelength. We know that an interferometer measures a phase shift to within a wavelength of the particles used. Since the de Broglie wavelength scales inversely with mass, the wavelength for a molecule will be $N$ times smaller than for any one of the constituent particles and the measurement precision will be improved by the same factor.

It is tempting to think that this means a molecule will allow us to do $N$ times better than a collection of individual atoms. However, this argument does not account for shot noise. It is true that, on a single measurement, a molecule allows a measurement precision $N$ times better than an atom. However, with atoms, we can make $N$ measurements with the same resources as a single molecule, which gives a shot noise improvement by a factor of $N^{1/2}$. So, overall the molecule wins by a factor of $N^{1/2}$ and gives a resolution that scales as $1/N$, which is the Heisenberg limit.

Superpositions of $C_{60}$ molecules have already been observed in the laboratory [29] and it is also possible to achieve cat-like superposition states without having to bind the particles together [30 – 33]. Interferometry with cat-like states has been observed in the laboratory for three [34] and six [35] beryllium ions and the phase enhancement of equation (12) has been confirmed. This suggests that the use of cat states may be a viable route to achieving the Heisenberg limit. So it would seem that we are done in our quest to achieve the ultimate precision limit.

Things, however, are not quite that simple. The problem with cat states is that they are incredibly fragile. This is one reason why we do not see superpositions of macroscopic objects in our everyday world. The basic idea is that, if a superposition state interacts with its environment, it can leave information in the environment about what state it was in. For example, the buckyball molecule passing through the interferometer could leave information in the environment that betrays which path it took. It is well known from Young's two-slit experiment that as soon as there is information about which path the particle took, the superposition is destroyed and no interference fringes are observed. This applies whether or not the path taken is actually measured, but only on whether it *could* be known in principle [36]. Macroscopic superpositions are particularly fragile since as soon as there is information about the path that any one particle took, the cover is immediately blown for all the rest of them too.

This does not mean that cat states cannot exist. If we are careful enough and clever enough we can mimimize the effects of the environment and so create cat states in the laboratory. Experiments have tackled this major challenge with remarkable success, creating ever-larger cats. However, when it comes to using these states for making precision measurements, Huelga *et al.* identified a major problem [37]. If we think about what we want for a good clock (or frequency standard) we want a system that can accurately distinguish small divisions of time (e.g. we want our clock to have a second hand) and we also want our clock to be stable over long periods of time. When we check whether our wristwatch is running fast or slow, for example, we compare it with a reference clock, wait some time, and then compare it again. The longer we wait between comparisons, the more accurate our measurement is. The same idea applies to fountain clocks, where we want the time of flight of the atoms to be as long as possible. Cat states are good for accurately distinguishing small time intervals, however, their fragility means that they fail miserably when it comes to making comparisons over long time intervals. It turns out that these two effects exactly cancel each other out. So our efforts to use cat states in precision measurements may well have met a brick wall.

## 7. Another way forward: number-correlated states

In order to overcome this problem, it is helpful to backtrack a little. A cat state was just our first attempt at finding a state with the maximum possible number uncertainty. It now seems that we have a stricter requirement: what we really want is a state that has the largest possible number uncertainty, and is also relatively robust to interactions with the environment. It turns out that just such a state is obtained if we send precisely the same number of particles into each input port of the interferometer [38–40].

Suppose we send exactly one particle into each input port. The initial state is $|\psi\rangle = |1\rangle|1\rangle$, and the state after the first beam splitter can be shown to be $|\psi\rangle = (|2\rangle|0\rangle + |0\rangle|2\rangle)/2^{1/2}$. This has a cat-like structure in that both particles emerge from one output or the other—there is no possibility of one particle emerging from each. This is known as the Hong–Ou–Mandel effect [41]. This result can readily be generalized to larger numbers of particles [39]. For $N/2$ particles at each input port, where $N$ is even, the state after the first beam splitter is

$$|N/2\rangle|N/2\rangle \rightarrow \frac{1}{(2^N)^{1/2}} \sum_{m=0}^{N/2} \frac{[(2m)!(N-2m)!]^{1/2}}{m!(N/2-m)!} |2m\rangle|N-2m\rangle. \tag{13}$$

This has a slightly unwieldy form and we do not need to worry about it too much. However, the probability distribution of the number of particles in either of the paths is plotted in figure 5 for this state with $N = 40$. The most notable feature of this distribution is that the number fluctuations are large. In fact the fluctuations are of the same size as the total number of particles in the system, $\Delta N \sim N$, which is just what we want. We now need to consider how well this state stands up in the presence of loss. It turns out that it does rather well.

Unlike a cat state, the distribution in figure 5 has a broad 'plateau' of values between $m = 0$ and $m = N$. This means that if we detect one particle on a particular path, this will not unambiguously tell us where all the other particles are. In other words, unlike a cat state, if information is left in the environment about which way a particle went, the cover is not blown for all the other particles. We would therefore expect this state to be more robust to the effects of interactions with the environment. Again this is just what we want. A more detailed analysis of the effects of loss has been carried out [42] and shows that this is indeed true. By using equal number states at the input ports we get a state that combines sharp phase resolution with long term stability. It seems that we really can have our cake and eat it.

However, we should not celebrate just yet. First we need to consider how the signal of the phase shift is read-out. If we use equation (3), the output signal for an equal-intensity

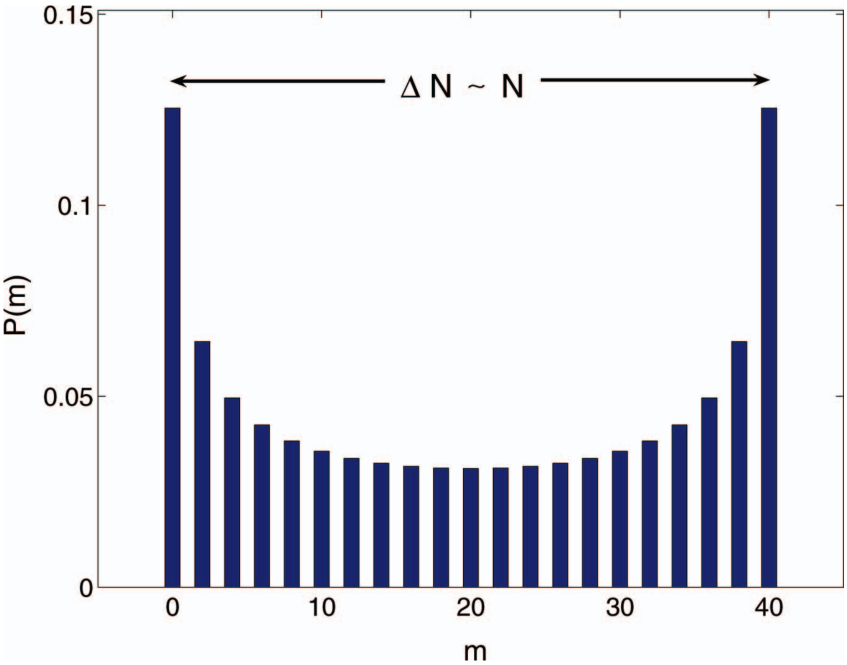


Figure 5. Probability distribution for the number of atoms at either output of a 50:50 beamsplitter if the input state was precisely 20 particles at each port.

input is $\langle n \rangle = 0$ independent of the phase $\phi$. This means that no signal of the phase is encoded on the mean population difference at the output ports. It turns out, instead, that the *variance* of the particles emerging from the outputs is related to $\phi$. This, in itself, should pose no major problem. It is certainly possible to determine the variance of the output distribution by repeating the experiment many times to build up the number distribution. From this, the phase shift should be able to be determined.

One problem that this technique throws up relates to the efficiency of the detectors. It has been shown that, to beat the shot-noise limit using this scheme, we would need detectors with efficiencies better than $1-1/N^{1/2}$ and, to reach the Heisenberg limit, we would need efficiencies better than $1-1/N$ [43]. This is a problem since, to take full advantage of the favourable number scaling, we would like the number to be as large as possible. For example, if we had four particles, this scheme could measure a phase with twice the precision of standard (shot-noise limited) interferometry, but if we had a million particles, it could do one thousand times better. However, as the particle number is increased, the required detector efficiency increases and, for any more than a handful of particles, this scheme is likely to be impractical [44]. This threatens to consign it to being little more than a theoretical curiosity.

However, one way we can overcome the problem of detector efficiencies is by making use of collapses and revivals of the relative phase of the output state [45]. We can understand this effect as follows. Suppose that the energy of a system depends nonlinearly on the particle number. One example of this would be if we had atoms that interact with one another. In this case, there will be some energy $U$ associated with the interaction between a pair of particles, and since every particle interacts with every other one, the total interaction energy for a collection of $N$ particles will be $UN(N-1)$, which is nonlinear. Now, each number eigenstate, $|m\rangle$, will evolve with time as $\exp(-iE_mt/\hbar)|m\rangle$, where $E_m$ is the total energy of $m$ particles. This means that the phase depends nonlinearly on the number of particles. Suppose now that we had a superposition of different numbers (such as in a coherent state). Every term in the superposition will evolve at a different rate and so the phase of the state will 'diffuse'. The broader the range of numbers in the superposition, the more rapidly this diffusion will take place. After some time, the phase will have completely smeared-out—this is known as a phase collapse. However, since particle number is discrete, if we wait a longer time, eventually all the phases will get back into step and we get a phase revival [46,47].

So how does all this help us with overcoming the problem of detector inefficiencies? Well, we saw earlier that, in this system, the phase shift is encoded on the number variance of the state at the output of the interferometer. Since the larger the variance is, the more rapidly the phase collapses, if we could measure this rate of collapse, we should be able recover information about the phase shift. In practice, this could be achieved as follows. The input state $|\psi\rangle = |N/2\rangle|N/2\rangle$ is first passed through an interferometer and the output is then allowed to undergo phase diffusion for some fixed time, $t$. Finally the particles are allowed to overlap and an interference pattern is detected between them. If there is a well-defined phase between the outputs (i.e. it has not diffused much) then a clear interference pattern would be observed. If in time, $t$, the phase completely collapses then no interference fringes would be observed on an ensemble of measurements. This means that the rate of collapse and hence the variance of the output (and value of $\phi$ that we want to measure) can be determined by how clear the fringes are—this is called the fringe visibility.

In figure 6 we have plotted an example of how the visibility of the interference fringes varies with $N\phi$ for a particular hold time, $t$. This hold time has been chosen to optimize the dependence of the contrast with the phase shift, $\phi$ [45]. We see that there is a clear series of 'lobes' that correspond to the collapses and revivals of the phase. The key feature is that these lobes have a width of approximately $N\Delta\phi \sim 1$. This means that by changing the phase by an amount of order $1/N$, the visibility changes from a local maximum (clear fringes) to zero (no fringes). This sensitivity should allow us to resolve phase shifts to within $1/N$, i.e. the Heisenberg limit. The appearance and disappearance of fringes is a dramatic observable that should be able to be seen in the laboratory. Furthermore, the beauty of this scheme is that it does not matter if we do not detect all the particles. We could even miss the majority of them and it would not matter—the fringes would still

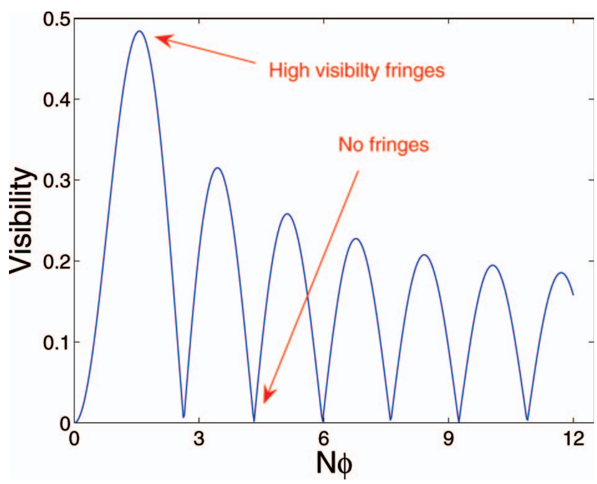


Figure 6. Plot of the visibility as a function of $N\phi$ of the interference fringes seen when particles are imaged after undergoing relative phase diffusion for the optimum hold time, $t$.

either be visible or not and that is all that concerns us. This is great news. It seems that now we really are done and have found a practical candidate for making measurements at the best possible precision allowed by quantum mechanics.

## 8. Conclusion

In this article, I hope to have conveyed some of the profound importance that measurements have in science. Measurements allow us to carry out the scientific process and anything that limits what we can measure will limit how far we can take science. Improving measurement technologies is crucial since it allows us to subject theories to increasing levels of scrutiny and so develop a better understanding of the physical world. It is also fascinating to turn this approach on its head and use new theories to make better measurements. One example is quantum theory, which allows more precise measurements than anything allowed by classical physics with the same resources. Measurements are also interesting for the important role they play in our everyday lives. The history of navigation, for example, is closely pinned to the history of accurate timekeeping right from the early days of exploration up to the modern satellite navigation systems we use in our cars.

One question that still remains from all this discussion might be 'Is the Heisenberg limit of quantum theory truly fundamental or can we do better?'. As far as we can tell at the moment, it would seem that it really is fundamental. At least we seem to be bound to it if we are unwilling to abandon quantum mechanics, which is the best physical theory we currently have. So maybe we really have found the ultimate limit to measurement precision and future developments in metrology will just be concerned with mopping up the details and finding more efficient implementations. But, then again, this viewpoint has more than an echo of Michelson's statement at the end of the nineteenth century that the future of physics lay in looking in the sixth place of decimals.

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