Phase resolution for Bose-Einstein condensates

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We discuss how the nature of the relative phase between two entangled condensates differs depending on whether the entanglement is established by measurements or by coupling. In the former case, the limit of the resulting phase resolution scales inversely with the square root of the number of atoms. This limit can be surpassed, however, when the condensates are entangled by coupling and can reach the fundamental Heisenberg limit where the phase resolution scales inversely with the number of atoms involved.

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Several recent papers have discussed how a relative phase can be established between Bose-Einstein condensates. One way this can be achieved is to measure the interference pattern between condensates in their region of overlap [1–3]. Another method, which has received recent attention, is to allow an exchange of atoms between condensates by coupling them [4]. These two methods are distinct, but both rely on creating an entanglement between the condensates. In this paper, we want to investigate whether these phase preparation methods can be distinguished by the nature of the phase of the final state.

Cirac *et al.* [5] have shown that the interference pattern measured between two condensates, initially in number states, cannot be distinguished from the interference pattern that would be measured if the two condensates were initially in coherent states. This suggests that the phase developed by such a measurement is ''classical'' in nature. We would like to investigate whether this is the case for condensates which have been entangled by coupling. In particular, we would like to see whether coupling can lead to better phase resolution than can be obtained by measurement.

Let us begin by considering the case of measurement. We consider two condensates, a and b, both initially in number states with n atoms, $|\psi_0\rangle=|n\rangle|n\rangle$. We allow atoms from the two traps to fall onto a detector and record the times at which atoms are detected [see Fig. 1(a)]. In the Heisenberg picture, when an atom is detected at time, t, the state, $|\psi_0\rangle$, is acted on by

$$C=\frac{1}{\sqrt{2}}(a\mathrm{e}^{-i\omega_a t}+b\mathrm{e}^{-i\omega_b t+i\phi}), \tag{1}$$

where ω_a and ω_b are the frequencies of traps a and b. The additional phase factor, $\phi=2\pi\delta/\lambda_{dB}$, depends on the path length difference, δ of the two traps from the detector, and the de Broglie wavelength of the atoms, λ_{dB}. For equal path lengths, ϕ would vanish. If the modes are degenerate, $\omega_a=\omega_b$, and we transform to a frame rotating at the same frequency, this can be written simply as $C=(a+b\mathrm{e}^{i\phi})/\sqrt{2}$. If n atoms are detected (i.e., half of them) the final state, $|\psi\rangle$, is given by

$$|\psi\rangle\propto\left[\frac{1}{\sqrt{2}}(a+b\mathrm{e}^{i\phi})\right]^n|n\rangle|n\rangle$$
$$\propto\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{n!\mathrm{e}^{i(n-m)\phi}}{\sqrt{(n-m-1)!(m-1)!}}|n-m\rangle|m\rangle. \tag{2}$$

As expected, this is an entangled state.

We would now like to examine the nature of the phase distribution of this state. To do this, we make use of the basis of states of well-defined phase [6],

$$|\theta_l\rangle=\frac{1}{\sqrt{s+1}}\sum_{p=0}^{s}\mathrm{e}^{ipl\epsilon}|p\rangle, \tag{3}$$

where $l=0,1,\ldots,s$, $\{|p\rangle:p=0,\ldots,s\}$ denotes the Fock states, and $\epsilon=2\pi/(s+1)$ is the rotation between adjacent phase states. The index s parametrizes the Hilbert space and, in general, we need to take the limit $s\to\infty$. As described

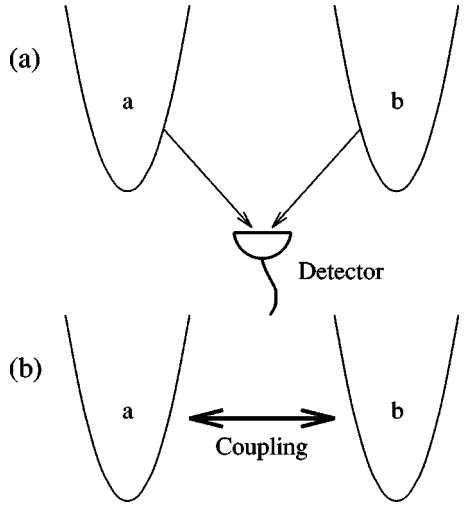

FIG. 1. Two methods of establishing a phase between the condensates. In (a), atoms from both condensates are allowed to fall onto a detector. An entanglement and relative phase arises since we cannot know from which condensate a detected atom has come. In (b), the two condensates are entangled by coupling them with Raman pulses for a time $t=\pi/4\Gamma$.

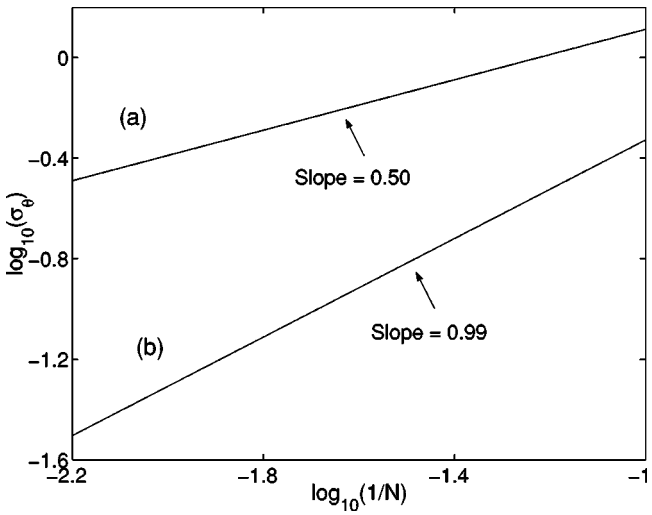

FIG. 2. Variation of the phase resolution, σ_θ, with total atom number, N, for (a) measurement-induced entanglement, and (b) entanglement due to coupling.

elsewhere [6], we should calculate the moments as a function of s and then take the limit $s\rightarrow\infty$, rather than the other way round. This is an important distinction to make for light states, for which the Hilbert space is infinite due to photons readily being created and destroyed. In our case, however, with a fixed finite number of atoms we do not need to take the infinite limit of s. We simply need our basis to extend over the total number of atoms in the state we are examining (2), i.e., $s=n$.

The probability that the relative phase between the two output modes is $\Delta\theta$, where $\Delta\theta$ is an integral multiple of ϵ, is given by calculating the overlap of the final state (2) with the joint phase state $|\theta_l\rangle|\theta_{l-\Delta\theta/\epsilon}\rangle$ and summing over all values of the absolute phase, which is parametrized by l [7]. A straightforward calculation of this distribution yields,

$$P(\Delta\theta) \propto \left|\sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\frac{n!}{\sqrt{(n-m-1)!(m-1)!}}e^{im(\Delta\theta-\phi)}\right|^2. \tag{4}$$

As expected, the path length difference, δ, simply shifts the phase of the distribution by $\phi=2\pi\delta/\lambda_{dB}$.

The function (4) can readily be calculated, allowing us to extract the dependence of the phase resolution on the total number of atoms, N, in the final state. We determine the full width at half maximum (FWHM), σ_θ, of the phase distribution (4) for a range of values of N and then plot $\log_{10}(\sigma_\theta)$ against $\log_{10}(1/N)$. The result, shown in Fig. 2(a), is a straight line with slope 0.50. This means that the phase resolution of the state varies with N as $\sigma_\theta\propto 1/\sqrt{N}$. This is the so-called ''standard limit'' and supports the discussion of Cirac *et al.* [5] as it is the same resolution that would be obtained if the two condensates were initially in coherent states.

With this as our benchmark, we would now like to see whether better phase resolution can be achieved when the condensates are entangled by coupling. As our guide, we turn to a closely related study of optical interferometry. The usual approach to interferometry, which uses coherent light as the input, cannot surpass the standard limit. However, it has been shown that by using correlated number states as the inputs, the phase resolution of a Mach-Zehnder interferometer can reach the Heisenberg limit, $\sigma_\theta\propto 1/N$ [7–9]. We take an analogous approach to create a Heisenberg limited relative phase between condensates.

In the optical case, if two identical number states are fed into the input ports of a 50:50 beam splitter, the relative phase between the two output modes is Heisenberg limited. The beam splitter can be considered to rotate the state in phase space in such a way that the phase fluctuations are transferred to the amplitude quadrature and vice versa [8]. A state with minimal amplitude fluctuations before the beam splitter will have minimal phase fluctuations afterwards.

Bouyer and Kasevich have shown that the operation of passing two photon states, a and b, through a 50:50 beam splitter is formally equivalent to coupling condensate states with resonant Raman pulses for time $t=\pi/4\Gamma$, where Γ is the coupling strength [10]. The unitary operator for this procedure is,

$$U=\exp\left[i\frac{\pi}{4}(ab^\dagger+a^\dagger b)\right]. \tag{5}$$

This suggests that for condensates we can follow a scheme similar to that for correlated optical interferometry, by using Raman coupling as the equivalent of the beam splitter [see Fig. 1(b)]. To make an accurate comparison with the measurement case, we start with the same state as before, i.e., two Fock states each with n atoms, $|\psi_0\rangle=|n\rangle|n\rangle$. We then turn on two laser fields which Raman couple the condensates for time $t=\pi/4\Gamma$. At the end of this coupling time, the entangled state is,

$$|\psi\rangle=\exp\left[i\frac{\pi}{4}(ab^\dagger+a^\dagger b)\right]|n\rangle|n\rangle \propto \sum_{m=0}^{n}\sqrt{\frac{(2(n-m))!}{(n-m)!^2}}\sqrt{\frac{(2m)!}{m!^2}}|2(n-m)\rangle|2m\rangle. \tag{6}$$

We would now like to study the phase distribution for this state and compare it with the result for measurement-induced entanglement. Making use of Eq. (6) and following the same technique as before, the relative phase distribution for this state is

$$P(\Delta\theta)\propto\left|\sum_{m=0}^{n}\sqrt{\frac{(2(n-m))!}{(n-m)!^2}}\sqrt{\frac{(2m)!}{m!^2}}e^{2im\Delta\theta}\right|^2. \tag{7}$$

The same result has been derived for photons passing through a beam splitter [7].

As before, we find the N-dependence of σ_θ by calculating this distribution for different values of N. A plot of $\log_{10}(\sigma_\theta)$ against $\log_{10}(1/N)$ is shown in Fig. 2(b). As was the case for the results shown in Fig. 2(a), this is a straight line which indicates that σ_θ has the same functional form as for the measurement case, $\Delta\theta \propto 1/N^r$. However, we immediately notice two differences from the result in 2(a). First, we see that the phase resolution is much better than for the measurement case: for 160 atoms, the resolution is roughly an order of magnitude better. Second, the slope of 2(b) is greater than 2(a), which indicates that σ_θ has a stronger N dependence. A fit through the points gives a value very close to $r=1$, which means that $\sigma_\theta \propto 1/N$, i.e., the phase is Heisenberg limited. This is an important result as it shows that we can achieve the fundamental limit of phase resolution (as governed by the uncertainty principle) by coupling condensates using Raman pulses.

To achieve the Heisenberg limit, it is important that the initial number states are correlated in order to minimize amplitude fluctuations. This may be able to be achieved by amplitude squeezing the initial joint state [11] or by creating correlated atom pairs by a process such as four-wave mixing [12].

Our starting state, $|\psi_0\rangle = |n\rangle|n\rangle$, is highly squeezed and nonclassical. We see from the results in the first part of this paper that, by measuring the phase of this state, we degrade the quality of the phase information that it can contain to the standard limit. In general, however, our initial state will not be squeezed but will qualitatively be like Eq. (2), i.e., it will be an entangled state which allows the total number of atoms to be fixed but the number in each condensate to be uncertain. In this latter case, the measurement process does not degrade the quality of the phase as it has already been brought down from the squeezed Heisenberg limit. Classical states of the form of Eq. (2) are attractors of phase measurements and so further measurements will not disrupt them. This tells us that if we start with a nonclassical squeezed state, we should perform the operations we want with this state (e.g., as the input to an interferometer) and extract information from it only at the end. As soon as measurements are made on the state, the useful phase information that it can contain is degraded until it reaches the classical level.

In summary, we have demonstrated that the nature of the relative phase between condensates depends on the method of entangling them. The entanglement that arises when we measure an interference pattern between condensates, leads to a phase resolution given by the classical standard limit, $\sigma_\theta \propto 1/\sqrt{N}$. When condensates are entangled by coupling, the nature of the phase is very different and the phase resolution can reach the fundamental Heisenberg limit, $\sigma_\theta \propto 1/N$. These Heisenberg limited states may have important consequences for applications such as interferometry and frequency standards where phase resolution is of utmost importance. They may also provide a valuable tool for investigating the nature of entanglement.

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