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Proposals for creating Schrödinger cat states in Bose–Einstein condensates

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Abstract. Three different schemes for producing Schrödinger cat states in Bose–Einstein condensates are outlined and the effects of loss in each of them compared. The first scheme involves coupled interacting condensates and proves to be very fragile to loss. This is improved upon with a second scheme which first evolves a cat state in phase space and then rapidly transforms it to a number cat state. Finally a third scheme is discussed which makes use of number correlated condensates and is remarkably robust to loss. It may prove to be valuable for experimentally creating such states.

1. Introduction

The theory of quantum mechanics, which successfully describes the physical world on a microscopic level, allows systems to exist in a coherent superposition of different states. When this is scaled up to macroscopic systems, however, an apparent contradiction arises. Our experience is that if a system has several macroscopically distinguishable states available, it will always be found in one of them. This anomaly was pointed out in the well-known Schrödinger cat paradox [1]: a thought experiment in which a cat is prepared in a superpostion of being alive and dead. Such a state defies our conception of the real world.

Despite this, under special condtions which carefully avoid decoherence [2], so-called 'Schrödinger cat' states can be produced in the laboratory. They have been demonstrated in a number of systems including the spatial coordinates of a single atom [3], the internal state of four ions [4], and the current in a superconductor [5]. A detailed experimental study of the effect of decoherence has also been carried out on a mesoscopic superposition of two optical coherent states [6].

A Bose condensate consists of a macroscopic number of particles in a single quantum state and so seems to be an ideal system for creating similar macroscopic superpositions. There have been some proposals for how approximate cat states may be produced [7–9], however there has been no direct observation of them.

Current cat state schemes consider creating a superposition of coherent states which are macroscopically distinct in their mean number or phase. In this paper, we discuss methods of creating superpositions of Fock states. In particular, we

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consider 'ideal' cat states, which are a coherent superposition of all the atoms being in one mode or all in the other. We can write this as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N\rangle|0\rangle + e^{i\eta}|0\rangle|N\rangle). \tag{1}$$

States of this form are of great theoretical interest since their number distribution has the maximum possible variance and consequently their phase resolution is optimal. This means that they may have important applications in interferometry [10], frequency standards [11, 12], and quantum information [13, 14]. They may also allow us to probe the boundary between quantum and classical physics and perform controlled studies of quantum measurement. The schemes we present are all relatively straightforward and should be possible with current technology.

We begin in section 2 by considering two coupled interacting condensates. We show that under the right conditions this system evolves towards a cat of the form (1), but is very fragile and is destroyed by the loss of even a single atom. In section 3, we discuss how a cat state may be created in a more robust fashion by first creating a cat in phase space and then transforming into the form (1). This is a considerable improvement over the first scheme since loss simply degrades the cat by some random amount rather than completely destroying it and the degree of degradation is independent of the rate of loss. Finally in section 4 we improve the robustness of the scheme even further by making use of number correlated condensates to show how number cat states may be reliably prepared even in the presence of large losses.

2. First scheme

In the first scheme, we study how the number distribution of two coupled interacting condensates evolves with time. We associate the two condensate modes with the annihilation operators $\hat{a}$ and $\hat{b}$. Mode a is initially in a number state, $|N\rangle$ and b is initially in the vacuum state, $|0\rangle$. We do not need to know the value of N, and so this is a readily available resource in the laboratory. We then couple these two modes with resonant Raman pulses to create a pair of condensates with a relative phase defined to the standard quantum limit. If this step is very fast compared with the timescale of the nonlinear evolution, we can ignore the effects of interactions and, after a quarter Raman cycle, the state is give by

$$\begin{aligned}|\psi\rangle &= e^{i\pi(\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a})/4}|N\rangle_a|0\rangle_b \\ &= \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\sqrt{\frac{N!}{k!(N-k)!}}e^{-i\pi k/2}|k\rangle_a|N-k\rangle_b.\end{aligned} \tag{2}$$

This step has already been experimentally demonstrated [15].

We would now like to evolve this distribution towards a cat state. To do this, we first apply a far off resonant light pulse to mode a to shift the phase by $\pi/2$ [8]. The state after this step is

$$|\psi\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\sqrt{\frac{N!}{k!(N-k)!}}|k\rangle_a|N-k\rangle_b, \tag{3}$$

and the two modes a and b have zero mean relative phase.

Next we couple the two condensate modes with resonant Raman pulses, which is equivalent to Josephson coupling the modes or allowing tunnelling through a barrier, and allow them to evolve under the influence of the nonlinear interactions. The Hamiltonian for this evolution is taken to be

$$H = U(\hat{a}^{\dagger 2}\hat{a}^2 + \hat{b}^{\dagger 2}\hat{b}^2) + \Gamma(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \tag{4}$$

where U is the interactomic interaction strength which we take to be the same for each mode for simplicity, and Γ is the coupling strength. We take the trap frequencies to be the same for the two modes and have removed them by transforming to a rotating frame. In reality, there would be some cross-interaction between the condensates [8]. However, for a weak overlap between the condensates, this term is much smaller than the interaction within each mode and, following other authors [16], we have neglected it. The inclusion of this term does not qualitatively change our results. The parameter U depends on the intrinsic atomic interactions and the shape of the traps. This can be tuned in principle [17], but for a given experiment is fixed. The coupling rate, Γ, can be controlled by the experimentalist by varying the strength of the coupling laser.

In figure 1, we have plotted how the relative number distribution of (3) evolves for the parameters $N = 100$, and $U/\Gamma = 0.02$. This last parameter has been chosen to give the optimum cat state. In figure 1 (*a*), we plot the variance of $N_a - N_b$ as a function of Γt. This is a measure of the width of the relative number distribution and we see that it increases to begin with, reaching a maximum at around $\Gamma t = 1.9$. The maximum is taken to indicate when the state is most cat-like.

In figure 1 (*b*), we plot the number distribution of mode a at this optimum time. The original number distribution is also shown as a dashed line for comparison. We see that the final state is a good approximation to an ideal cat state.

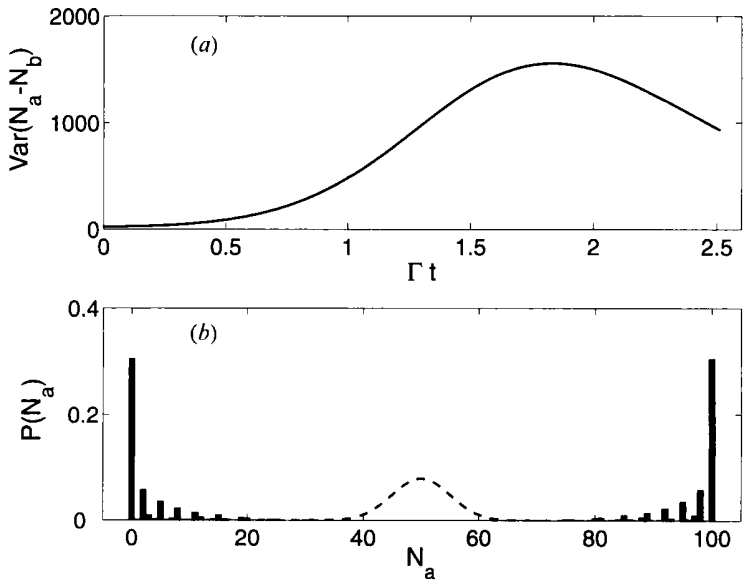

Figure 1. (*a*) Variance of $(N_a - N_b)$ as a function of Γt. (*b*) Number distribution of mode a at the optimum cat time (bar graph) compared with the initial number distribution (broken curve).

2.1. *Semiclassical model*

In order to understand these results, we perform a semiclassical analysis of the system. Following the same formalism as [18], we define the quantities of fractional population imbalance,

$$z(t) \equiv \frac{\langle \hat{b}^\dagger \hat{b} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle}{\langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle} = \frac{N_b(t) - N_a(t)}{N}, \tag{5}$$

where $N = N_a + N_b$ is the constant total number of atoms, and relative phase

$$\phi(t) \equiv \arg\{\langle \hat{a}^\dagger \hat{b} \rangle\} = \theta_b(t) - \theta_a(t). \tag{6}$$

We can derive semiclassical equations of motion for this system in terms of these new quantities [19]

$$\dot{z}(t) = -2\Gamma\sqrt{1 - z^2(t)}\sin(\phi(t)) \tag{7}$$

$$\dot{\phi}(t) = -2UNz(t) + \frac{2(t)}{\sqrt{1 - z^2(t)}}\cos(\phi(t)) \tag{8}$$

Numerical solutions of (7) and (8) for different initial conditions are shown in figure 2. We set the parameters to be the same as for the quantum calculation, $U/\Gamma = 0.02$ and $N = 100$. Each trajectory corresponds to $z(0) \in \{-0.12, -0.08, -0.04, 0, 0.04, 0.05, 0.12\}$ and, since the initial state has zero relative phase between the modes (see equation (3)), we take $\phi(0) = 0$ for each trajectory.

The quantum state can be thought of loosely as a superposition of these classical realizations. Here we interpret each trajectory to represent part of the state and, in particular, a few adjacent terms in (3), i.e. terms with similar numbers of atoms in mode a. The way that the state is split up into trajectories is arbitrary and does not affect our results. The parameters z and ϕ now respectively represent the mean number difference between the modes and the mean phase for each of

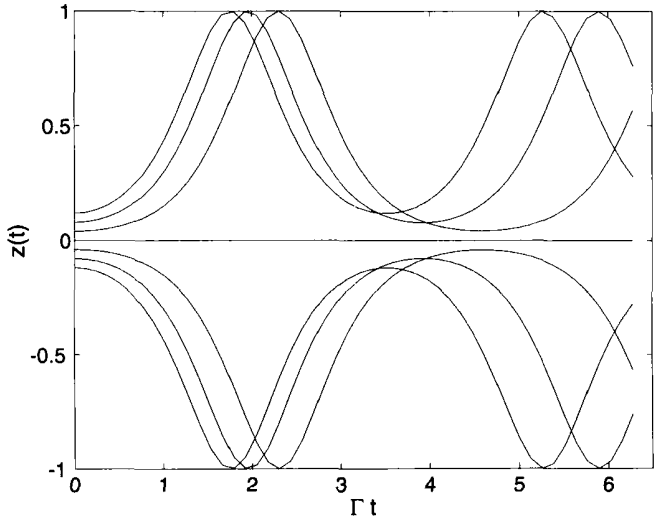

Figure 2. Solutions for $z(t)$ of the semiclassical equations for the initial conditions $\phi = 0$ and $z(0) \in \{-0.12, -0.08, -0.04, 0, 0.04, 0.08, 0.12\}$.

these parts. The evolution of the whole state can be seen by observing the evolution of each of its parts or trajectories.

We see that the trajectories oscillate and, in particular, diverge into two distinct groups. At approximately $\Gamma t = 2$, the distribution has evolved to the optimal macroscopically distinct superposition. This is in good agreement with the quantum calculation.

Identical results would be achieved for attractive interactions in the condensates and an initial phase difference of $\phi = \pi$ between the modes. The possibility of creating cat states by coupling condensates with attractive interactions has been mentioned by previous authors [20].

2.2. *Frequency standard*

Huelga *et al.* [12] have investigated how cat states may be used to improve frequency standards. In their scheme they considered creating a cat state using a theoretical control-not (CNOT) gate. They then allowed this state to evolve freely for some time before disentangling the cat with another CNOT gate.

They showed that the final population of the two modes oscillates with a frequency enhanced by a factor of N for cat states and pointed out that this should allow for greatly enhanced resolution of the difference in frequencies of the two modes over the free evolution time. In this section, we show how this scheme may be implemented.

Our scheme for creating cats described above is equivalent to the quantum gate process for generating cats. It makes the transformation

$$|N\rangle|0\rangle \rightarrow \frac{1}{\sqrt{2}}(i|N\rangle|0\rangle + |0\rangle|N\rangle) \tag{9}$$

$$|0\rangle|N\rangle \rightarrow \frac{1}{\sqrt{2}}(i|0\rangle|N\rangle + |N\rangle|0\rangle). \tag{10}$$

If we create a cat by this method (say of the form (9)) and then allow the state to evolve for time, t, where the two modes experience different frequencies, the new state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(i\,\mathrm{e}^{-iN\omega t}|N\rangle|0\rangle + |0\rangle|N\rangle), \tag{11}$$

where ω is the difference in mode frequencies. Since our method (apart from any loss) is deterministic, we can disentangle the modes simply by reversing the creation scheme. The sign of the nonlinear interaction needs to be reversed which may be achieved by adjusting the magnetic fields [17] and a phase shift of π needs to be introduced to one of the modes. As we have seen, this can be achieved using far detuned light pulses. With these two changes, the sign of the right hand side of equations (7) and (8) is reversed and the state is effectively propagated back in time.

The state after this disentangling procedure is now given by transforming (11) by the inverse of (9) and (10). This gives, apart from an irrelevant global phase factor,

$$|\psi\rangle = \cos\left(\frac{N\omega t}{2}\right)|N\rangle|0\rangle + \sin\left(\frac{N\omega t}{2}\right)|0\rangle|N\rangle. \tag{12}$$

This is still a superposition of two macroscopically distinct states, but the amplitudes of the two possible outcomes differ and depend on the frequency difference of the modes during the free evolution. Importantly, this frequency is enhanced by a factor N, which is potentially very large.

If we were to measure which mode the atoms are in (they should all be in one or all in the other), the probability they are in mode 1 is

$$P_1 = \tfrac{1}{2}(1 + \cos(N\omega t)). \tag{13}$$

We see that the frequency of oscillations is enhanced by a factor of N. Measuring this for different values of t allows one to determine ω much more accurately than for coherent states. This could therefore be used as a very precise way of comparing frequencies. The N-fold enhancement of the population oscillations may also prove to be a clear signature of the presence of a cat state. This means that it may also be able to confirm that the method we have outlined does indeed produce cat states, and not just statistical mixtures of the two outcomes.

2.3. *Effects of loss*

The main enemy of cat states is decoherence [2] and is the reason we do not see macroscopic superpositions in everyday life. It is inevitable that any decoherence will destroy the final cat state. However, here we would like to consider the effect of loss during the preparation of the cat.

It turns out that when we include even very weak loss during the preparation scheme outlined above, the cat state is destroyed. We can understand this by considering the effect of loss on the state when it is nearing the end of its evolution time and is therefore close to a cat state. The action of loss from mode a is then,

$$\hat{a}(|N\rangle|0\rangle + |0\rangle|N\rangle)/\sqrt{2} = \sqrt{\frac{N}{2}}|N-1\rangle|0\rangle, \tag{14}$$

and similarly for loss from mode b. In each case, the loss of a single atom completely destroys the superposition.

This means that this scheme is useful only for cases where the evolution time is sufficiently short that on average no atoms are lost. We can write this condition as,

$$N \ll \frac{\Gamma}{2\kappa}, \tag{15}$$

where κ is the rate of loss per atom per unit time. In practice, this will give an upper limit to the number of atoms in a cat state formed by this method. In the next section, we discuss an alternative scheme which is more robust to loss.

3. Second scheme

The limitations of the first scheme stem from the fact that the number distribution evolves towards a cat state. This means that the state becomes increasingly fragile during its preparation. The idea of the second scheme is to evolve the *phase* distribution towards a cat state and then to rapidly transform this to a number cat state at the end. A macroscopic superposition of phase states is not so strongly affected by loss or by detecting atoms from a particular mode. This means it should be more robust during the evolution time.

Our technique is based on the observation that the output from a interferometer can be controlled by varying the relative path length of the two arms. In particular, it depends on the fact that, by introducing a phase shift to one arm, we can arrange for all the atoms (or photons) to emerge from one output port or, conversely, for them all to emerge from the other. Before outlining the cat state scheme, it is worthwhile discussing this result.

3.1. *Interferometer scheme*

In figure 3, we show an interferometer scheme. We consider the input at a to be a number state, $|N\rangle$, and the input at b to be a vacuum, $|0\rangle$. We will see later that, as in the first scheme, we do not need to know the value of N for the scheme to work. It is convenient to use the well-known Schwinger representation of the twin beam system through the following operators [16, 21],

$$J_x = \tfrac{1}{2}(\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b}) \tag{16}$$

$$J_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}) \tag{17}$$

$$J_z = \tfrac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \tag{18}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of modes a and b respectively. These satisfy the commutation relations

$$[J_x, J_y] = iJ_z \quad [J_y, J_z] = iJ_x \quad [J_z, J_x] = iJ_y. \tag{19}$$

The first step of the scheme is to pass the initial state through a 50:50 beam splitter. The analogue of this for atoms is to Raman couple the two modes resonantly for a time $t = \pi/4\Gamma$, where Γ is the coupling strength. These two operations are formally identical [22] and we will refer to them interchangeably. The operator for this step is $\exp(i\pi J_x/2)$.

This introduces a relative phase of $\pi/2$ between the arms of the interferometer. We wish to keep track of the relative phase throughout, and so we define the phase of the lower arm relative to the upper arm as $\Delta\theta$. After this step $\Delta\theta = -\pi/2$.

Next we introduce a phase shift, ϕ, to mode a. The operator for this step is $\exp(i\phi\hat{a}^\dagger\hat{a})$. However, since we are considering a case where the total number of atoms in the two arms is fixed, we can write this (apart from an irrelevant global

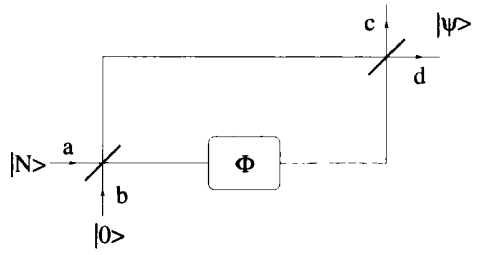

Figure 3. The interferometry scheme. An input state $|N\rangle$ is coupled to a vacuum, $|0\rangle$, one arm is subjected to a phase shift, ϕ, and then the two components are recombined. The output state is $|\psi\rangle$.

phase factor) as $\exp[i\phi(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2] = \exp(i\phi J_z)$. The relative phase is now, $\Delta\theta = \phi - \pi/2$.

The final step of the interferometer is to recombine the two components with an operation identical to the first step. We can now draw all these steps together and write the full operation of the interferometer as

$$e^{i\pi J_x/2}\, e^{i\phi J_z}\, e^{i\pi J_x/2}|N\rangle_a|0\rangle_b = e^{i\phi J_y}\, e^{i\pi J_x}|N\rangle_a|0\rangle_b$$
$$= -e^{i\phi J_y}|0\rangle_a|N\rangle_b. \tag{20}$$

This last step follows since $\exp(i\pi J_x)$ corresponds to a half-cycle Raman pulse which coherently transfers all the population from one mode to the other.

If no phase shift is introduced, $\phi = 0$, (i.e. the relative phase of the two arms entering the second beam splitter is $\Delta\theta = -\pi/2$), the output state from the interferometer is

$$|\psi\rangle = -|0\rangle_c|N\rangle_d, \tag{21}$$

and if $\phi = \pi$, (i.e. $\Delta\theta = \pi/2$) the output state is

$$|\psi\rangle = -e^{i\pi J_y}|0\rangle_a|N\rangle_b$$
$$= -e^{-i\pi J_z/2}\, e^{i\pi J_x}\, e^{i\pi J_z/2}|0\rangle_a|N\rangle_b. \tag{22}$$

The last step can be checked using the operator theorem [23]

$$e^{\xi B}\, e^{A}\, e^{-\xi B} = \exp\left(A + \xi[B, A] + \frac{1}{2!}\xi^2[B, [B, A]] + \cdots\right). \tag{23}$$

Operating on the state, (22) becomes

$$|\psi\rangle = -|N\rangle_c|0\rangle_d. \tag{24}$$

Comparing (21) and (24) we see that, by varying the relative path length of the two arms, we can arrange for all the atoms to emerge from port c or for them all to emerge from port d.

In order to create our desired Schrödinger cat state (1), we wish to form a superposition of these two outcomes. This suggests a possible way to proceed. If we were able to create a state entering the second beam splitter that is a superposition of the two relative phases $\Delta\theta = -\pi/2$ and $\Delta\theta = \pi/2$, the output should be a superposition of the two outcomes (21) and (24), i.e. it should have precisely the form of (1).

Yurke and Stoler have studied such phase superpositions [24]. In particular, they have shown that an optical coherent state passing through an amplitude dispersive medium can evolve into a superposition of two coherent states π out of phase with one another. We would like to form an analogous state for Bose condensates. Conveniently, in condensates the dispersion arises naturally and is due to the nonlinear interactions between atoms. A coherent state of a condensate will undergo collapses and revivals of the phase due to this nonlinearity [25] and midway between these revival times, the state will be in a superposition of precisely the form we want.

We now draw all these ideas together into a practical proposal for creating Schrödinger cat states of the form of (1).

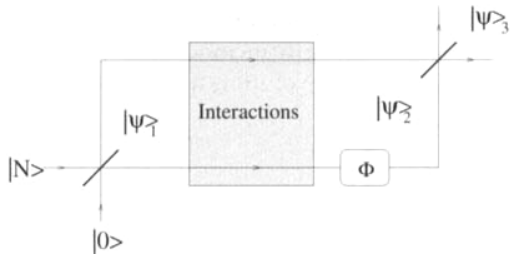

Figure 4. The second scheme for creating Schrödinger cat states with Bose condensates.

3.2. *Schrödinger cat scheme*

The proposal is depicted schematically in figure 4. Our initial state is a Bose condensate in a number state, $|N\rangle$. This is a readily available resource in the laboratory since we do not need to know the number of atoms in the state.

The first step is to couple the condensate to a vacuum mode with resonant Raman pulses. As discussed above, if the modes are coupled for a time $t = \pi/4\Gamma$, where Γ is the coupling strength, this is mathematically equivalent to the operation of a 50:50 beam splitter, and so we depict it as one in figure 4.

We take this step to be very fast compared to the timescale of the evolution due to the nonlinear interaction. This is given by the condition, $\Gamma \gg UN$, where U is the strength of the nonlinear interaction. This means that we can ignore interactions during the coupling and the state after this step is [8]

$$|\psi\rangle_1 = \mathrm{e}^{\mathrm{i}\pi J_x/2}|N\rangle|0\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\binom{N}{k}^{1/2}\mathrm{e}^{-\mathrm{i}\pi k/2}|k\rangle|N-k\rangle. \tag{25}$$

Hall *et al.* have already experimentally demonstrated a complete condensate interferometer scheme which is very similar to our proposal [15]. This is very encouraging for the practical feasility of our scheme.

We now wish to create a superposition of the relative phases $\Delta\theta = \pm\pi/2$ for this state. Following the work of Yurke and Stoler discussed above, we can achieve this by allowing the state to evolve naturally under the influence of its interatomic interactions for time $t = \pi/4U$, where U is the interaction strength. To begin with, we consider that U has the same value for each arm of the interferometer and that there are no interactions between the two arms. It turns out that this gives a superposition of two phase states as required. However, this superposition is either of $\Delta\theta = 0, \pi$ or $\Delta\theta = -\pi/2, \pi/2$ depending on whether the total number of atoms in the two arms is even or odd. This means that we would need to know whether the number is even or odd so that we could introduce an appropriate phase shift to create the state with $\Delta\theta = \pm\pi/2$. It is not clear how this could be done experimentally. We need to find a method that creates the correct superposition independent of the total number of atoms.

We can achieve this by considering different interaction strengths in the two arms. Cornish *et al.* have shown how magnetic fields may be used to tune the interaction strength over several orders of magnitude [17]. This would allow the interaction strength of the two arms to be adjusted to the required values and may

be important in a practical implementation of this step. In particular, we consider the case where the interaction strength in mode b is an integer multiple of the strength in a, U. The state after an evolution time of $t = \pi/2U$ is

$$|\psi\rangle_2 = \exp\left[\mathrm{i}\frac{\pi}{2}(\hat{a}^{\dagger 2}\hat{a}^2 + m\hat{b}^{\dagger 2}\hat{b}^2)\right]|\psi\rangle_1, \tag{26}$$

where m is an integer. If m is even, $m \in \{\ldots, -2, 0, 2\ldots\}$, and we apply a far detuned light pulse to advance the phase of one arm by $\pi/2$ as in the first scheme, the state can be written as

$$|\psi\rangle_2 = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\binom{N}{k}^{1/2} \mathrm{e}^{\mathrm{i}\pi[(m+1)k^2+(m-1)k]/2}|k\rangle|N-k\rangle. \tag{27}$$

We see that the relative phase is no longer N-dependent. For convenience, we set $m = 0$ in the remainder of this paper, though identical results hold for any even value of m.

Let us now take a moment to examine the phase distribution of (27) and to check that it is a superposition of two relative phase values. The relative phase distribution, $P(\Delta\theta)$, of the state $|\psi\rangle_2$ is given by

$$P(\Delta\theta) = |\langle\theta_{l-\Delta\theta/\varepsilon}, \theta_l|\psi\rangle_2|^2, \tag{28}$$

where $|\theta_l\rangle$ are the states of well defined phase [26], which take the form,

$$|\theta_l\rangle = \frac{1}{\sqrt{N+1}}\sum_{p=0}^{N} \mathrm{e}^{\mathrm{i}pl\varepsilon}|p\rangle, \tag{29}$$

where $\{|p\rangle : p = 0, \ldots, N\}$ denotes the Fock states, and $\varepsilon = 2\pi/(N+1)$ is the rotation between adjacent phase states.

The phase distribution of this state is shown in figure 5 for $N = 20$. We see that it is a superposition of the two relative phases $\Delta\theta = -\pi/2, \pi/2$ as required. This

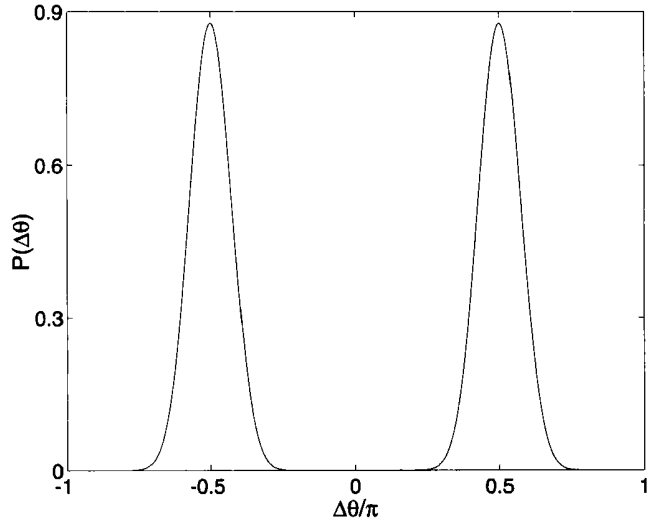

Figure 5. Plot of the phase distribution of $|\psi\rangle_2$ for $N = 20$.

form of the state holds for all values of N which is a very pleasing feature since no knowledge of N is required.

Finally, we complete the interferometry scheme by recombining the two components. This should transform the phase cat into a number cat and is done with Raman pulses identical to those used in the first step. Once again, this is performed much faster than the nonlinear evolution. The final state is then

$$|\psi\rangle_3 = \mathrm{e}^{\mathrm{i}\pi J_x/2}|\psi\rangle_2. \tag{30}$$

After some algebra, this can be written as

$$|\psi\rangle_3 = \sum_{r=0}^{N} C_r|r\rangle|N-r\rangle, \tag{31}$$

where

$$C_r = \sqrt{r!(N-r)!}\sum_{s=0}^{N}\sum_{t=t_1}^{t_2}\binom{N}{s}\binom{s}{t}\binom{N-s}{r-t}\mathrm{e}^{\mathrm{i}\pi(s^2+r-2t)}, \tag{32}$$

and $t_1 = \max(0, r+s-n)$ and $t_2 = \min(r,s)$. A plot of these coefficients for $N = 20$ is shown in figure 6. We see that the coefficients all vanish apart from C_0 and C_{20}. The final state is then of precisely the form we want,

$$|\psi\rangle_3 = \frac{1}{\sqrt{2}}(|20\rangle|0\rangle + \mathrm{e}^{\mathrm{i}\eta}|0\rangle|20\rangle), \tag{33}$$

where η is an unimportant phase. In fact, this process will work for any value of N, giving (1).

Comparing figure 6 with figure 1 (*b*), we see that this scheme is already an improvement over the first scheme since all but the maximum number difference coefficients vanish. We would now like to compare how the two schemes differ in the presence of loss.

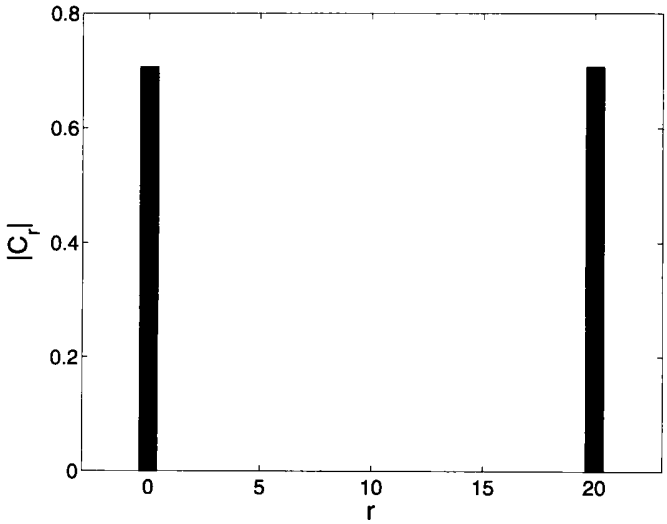

Figure 6. Plot of the modulus of the coefficients, C_r of the final state, $|\psi\rangle_3$ for $N = 20$. We see that the only non-vanishing coefficients are C_0 and C_{20}.

3.3. *Effects of loss*

It is straightforward to investigate the effect of the loss of an atom from (say) mode a during the interferometry scheme. Let us say an atom is lost at time ft, where $0 \leqslant f \leqslant 1$ is the fraction of the total nonlinear interaction time, t, at which the loss occurred. The state at the end of the interaction time is then given by

$$|\psi\rangle_2 = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{\tilde{N}} \binom{\tilde{N}}{k}^{1/2} e^{i\pi[k(k-1)]/2} e^{i\pi fk} |k\rangle |\tilde{N} - k\rangle, \tag{34}$$

where $\tilde{N} = N - 1$. We notice that, apart from the obvious fact that there is one atom fewer in this system, the only difference is the second exponential factor. This factor simply shifts the relative phases of the modes by the stochastic amount $f\pi$. The system is still in a superposition of two relative phases differing by π. However, both these relative phases are shifted by the same random amount between 0 and π.

If we consider the case $f = 0$, there is no shift of the relative phases and similarly for $f = 1$. This means that if there is any loss between the beam splitters either before or after the nonlinear interaction, it will have no effect on the cat state. It is only loss during the nonlinear interaction that introduces randomness.

The random phase shift is readily generalized to multiple losses. If n atoms are lost at respective fractions of the total interaction time, $f_1, f_2, \ldots, f_n$, the relative phase shift is $\pi(f_1 + f_2 + \cdots + f_n)$. Since the relative phase is a superposition of two values differing by π, this phase shift can be recast in the form $\pi\tilde{f}$, where $0 \leqslant \tilde{f} \leqslant 1$. In other words, the effect of many losses is the same as the effect of one.

The problem with a shift of the relative phase is that it degrades the quality of the cat state. As we have seen, a perfect cat state is only obtained when the state just before the second beam splitter is a superposition of $\Delta\theta = \pm\pi/2$. For $\Delta\theta = 0, \pi$ the number distribution of the output is singly-peaked and so not cat-like at all. For all values of the relative phase between these two extreme cases, the output has varying degrees of 'cattiness'. If we write the two relative phases as $\Delta\theta = \eta, \pi + \eta$, two macroscopically distinct peaks in the number distribution are clear for $\eta \approx 0.1\pi - 0.5\pi$. This means that about 80% of trials will produce a cat-like state regardless of how much loss there is during the preparation.

This scheme is therefore an improvement over the first scheme which couldn't tolerate any loss. To produce ideal cat states consistently, however, no loss can be supported. As before, this places a restriction on the maximum size of cat states that can be produced by this method. In the next section, we present a method to reliably produce cat states even when relatively large losses are present.

4. Third scheme

In the second scheme, we saw that loss didn't destroy the superposition of relative phases, but only introduced random shifts. It is the fact that the phase evolved nonlinearly with atom number that these random shifts arose. If we could find a way of creating relative phase superpositions without requiring the nonlinear evolution, atom loss should not destroy our method.

It turns out that relative phase superpositions of precisely the form we want may be generated by resonantly Raman coupling number correlated condensates, $|\psi\rangle = |N/2\rangle|N/2\rangle$. The phase resolution properties of number correlated

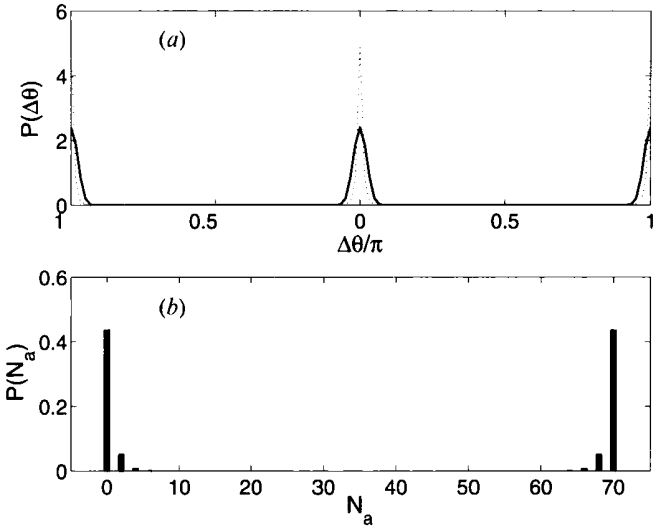

Figure 7. (*a*) Relative phase distribution for the optimally number squeezed state with $N = 100$ (dotted curve) and the phase distribution after the loss of 30 atoms (solid curve). (*b*) The final number distribution of mode *a* after phase shifting and Raman coupling. Despite significant loss, this is still a good approximation to a cat state.

condensates have been studied elsewhere [22, 27]. Here we are more interested in the value of the relative phase. If we Raman couple the two condensate modes for a quarter cycle, the relative phase distribution (as given by equation (28)) consists of a superposition of peaks at $\Delta\theta = 0, \pm\pi$, which is precisely the result we want.

It is promising that a scheme for generating good approximations to number-correlated condensates has already been described [18]. We use a state generated by this method as our starting point. This gives a more accurate assessment of the feasibility of cat state generation than using perfectly number correlated condensates. We take each mode to have a mean number of 50 atoms and use the optimally number squeezed state as our starting point. As can be seen in [18], the variance of the relative number for this state is approximately 1. The relative phase distribution of this state, after resonantly Raman coupling the modes for a quarter cycle, is shown as the dotted curve in figure 7 (*a*). As discussed above, we see that this consists of a superposition of peaks at $\Delta\theta = 0, \pm\pi$.

In order to create a cat state we require a state of the form (27), i.e. the number distribution should be Gaussian and the relative phase peaks should be centred at $\Delta\theta = \pm\pi$. The current state has a number distribution which is very flat [22, 27]. We can transform this to the required Gaussian simply by allowing loss from both modes. After the loss of a few times $\sqrt{N}$ atoms, the number distribution has the required form. The peaks in the relative phase distribution also broaden as a natural consequence of this loss. The solid curve in figure 7 (*a*) shows the relative phase distribution of the state after $3\sqrt{N} = 30$ atoms have been lost. In this case 16 were lost from mode *a* and 14 from mode *b*.

Unlike the method outlined in the previous section, loss doesn't introduce a random shift in the position of these peaks, it simply broadens them. The reason there is no phase shift is that, in this case, there is no nonlinearity present

during the evolution. This is probably unrealistic, there will always be some interatomic interactions. We now consider how small these need to be for this scheme to work.

If we consider that there is some small nonlinearity of strength δU, the evolution operator of the state is

$$\exp[\mathrm{i}\,\delta U t(\hat{a}^{\dagger 2}\hat{a}^2)]. \tag{35}$$

As we have seen above, the time of evolution, t, is the time required for $\sim\sqrt{N}$ atoms to be lost from the system. For large N, this is given by

$$t = \frac{1}{\gamma\sqrt{N}}, \tag{36}$$

where γ is the loss rate. If we now consider the loss of a single atom at a random fraction $0 \leqslant f \leqslant 1$ of this evolution time, the phase shift introduced is $\Delta\phi = 2\delta U f/\gamma\sqrt{N}$. However, we need to consider the phase shift introduced by the loss of $\sqrt{N}$ atoms. This has an upper bound of $|\Delta\phi| = 2\delta U/\gamma$, though in practice for large N it will be much less than this since all the phases will not add constructively. If we consider the distribution of phase shifts due to the loss of $\sqrt{N}$ atoms, it will be centred on $\Delta\phi = 0$ and have a width of $N^{1/4}$. So, we can reasonably take

$$|\Delta\phi| \approx \frac{4\delta U}{\gamma N^{1/4}}. \tag{37}$$

This gives us a condition on the result that the relative phase peaks are not significantly shifted by the loss. This will be the case for all N if $\delta U/\gamma \ll 1$. This should be able to be arranged experimentally and it is pleasing to note that this condition becomes less stringent for large N, e.g. for $N > 10\,000$, δU and γ can be comparable in size.

The final step in generating the required relative phase distribution is to apply a far-detuned light pulse to one mode to shift the phase by $\pi/2$. The phase distribution is now very similar to the one shown in figure 5 with the two peaks centred at $\Delta\theta = \pm\pi/2$.

The cat state can now be produced, as in section 3.2, by Raman coupling the modes for a quarter Raman cycle. The number distribution of mode a after this step is shown in figure 7 (*b*). We see that this is a very good approximation to an ideal cat state even though there were large losses present. This demonstrates a method of generating cat states which is not destroyed by the presence of loss, but actually *requires* loss to work.

The natural question to ask next is whether the amount of loss needs to be carefully controlled. In figure 8 (*a*), we show the final number distribution of mode a for the case where $5\sqrt{N} = 50$ atoms were lost. Again we see that this is a good approximation to an ideal cat state. We conclude that this method is not sensitive to the amount of loss so long as it is greater than about $2\sqrt{N}$ and the rate of loss from each mode is the same.

For completeness, we ask how well the initial state needs to be squeezed for successful generation of cat states. This question comes in two parts: first, how is the cat state affected if the initial state is not optimally squeezed, and second, what happens to the cat state if loss is present during the preparation of the number-correlated pair. In figure 8 (*b*) we show the final number distribution of mode a for

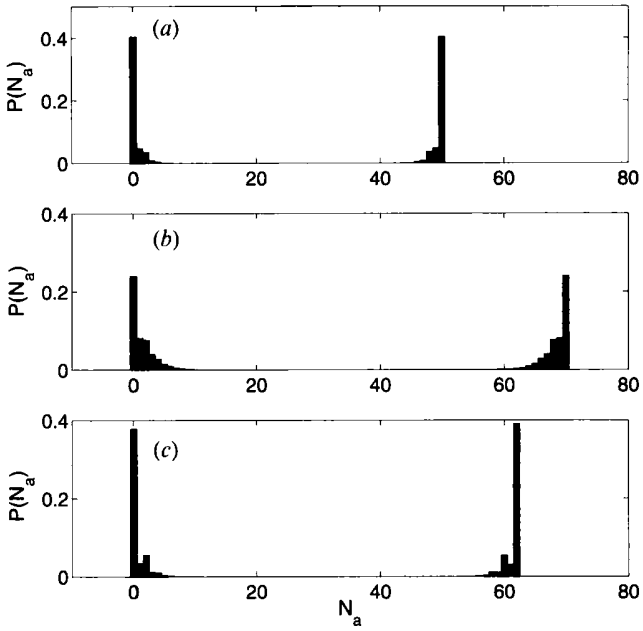

Figure 8. The final number distributions for mode a after the cat state generation scheme for (a) loss of 50 atoms after squeezing; (b) squeezing the initial state for only $1/\sqrt{2}$ of the optimum time and then loss of 30 atoms; (c) loss of 8% of the atoms during the squeezing time, then loss of a further 30 atoms afterwards.

loss of 30 atoms (i.e. the same as in figure 7), but where the initial state is not optimally squeezed. In this case the initial state was allowed to be squeezed for time $t = t_{\text{opt}}/\sqrt{2}$, where t_{opt} is the optimum squeezing time as discussed in [18]. This corresponds to underestimating the number of atoms in the original condensate by a factor of two. We see that, although degraded relative to figure 7 (b), the final state is still a good approximation to an ideal cat state.

In figure 8 (c), we show the final number distribution of mode a for loss of 30 atoms (i.e. the same as in figure 7 (b)), but where there is some loss during the preparation of the initial number squeezed pair. For the case shown here, 8% of the atoms were lost during the squeezing. We see that the final state is still a good approximation to an ideal cat state even though there is loss in all stages of its preparation. This last result is not surprising since it has been shown that the squeezing procedure is relatively robust to loss [18].

Overall this third scheme is only weakly sensitive to loss and is a great improvement over both the first and second schemes outlined. Furthermore, the fact that it is relatively insensitive to our knowledge of the number of atoms in the original system may make this scheme a strong candidate for the experimental generation of cat states in BEC's.

5. Conclusions

We have demonstrated three methods by which Schrödinger cat states may be generated and have compared their robustness in the presence of loss.

The first scheme involves coupling two interacting condensates. The trick is to get the right balance between the interaction strength and the rate of coupling. While straightforward, this method is very delicate and can be destroyed by the loss of a single atom. Its fragility does not rule it out as a practical method for producing cat states, but the number of atoms in a state created by this scheme will be limited.

The second scheme overcomes some of the difficulties of the first by evolving a cat state in relative phase space and then rapidly transforming it to a number cat state at the end. In this case, loss introduces a random shift to the cat in phase space. This shift manifests itself as a random degradation of the final number cat. In most trials, the degraded state is still 'cat-like' and, interestingly, the degradation is independent of the amount of loss. This second scheme is therefore an improvement over the first scheme in that it can tolerate loss provided that we are not concerned about the quality of the final cat.

The final scheme discussed not only tolerates loss, but requires it. It allows number cat states to be created reliably even in the presence of large losses. The success of this scheme lies in beginning by creating number–correlated pairs of condensates. These are more information-rich than cat states thereby providing the possibility of creating cat states from them even when information has been lost due to dissipation. Such a method may allow for the creation of larger cat states than the other schemes and could prove to be a useful method for producing cat states in the laboratory.

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