

Phase measurement of Bose-Einstein condensates in lattices

J. A. Dunningham

Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom

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We discuss how the phase distribution of a Bose-Einstein condensate in an optical lattice may be measured directly using existing experimental techniques. We also demonstrate how a modification of this scheme may be able to resolve states, that are close to the Mott insulator state, much more accurately than present methods. This may be of interest for analyzing the quantum state of condensates in recent experiments.

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Bose-Einstein condensates in optical lattices have recently been a subject of much theoretical and experimental interest. There have been proposals to use them for creating quantum logic gates [1], macroscopic quantum superpositions [2], and to achieve Heisenberg-limited resolution in interferometers [3,4]. Recent experiments have also demonstrated number squeezing between lattice sites [5] and even the phase transition from the superfluid to the Mott insulator (MI) regime [6].

Each of these experiments and proposals require a method for analyzing the quantum state that is created. In the first half of this paper we outline how such a measurement may be realized using existing experimental methods. The technique we propose is an atomic analog of the phase measurement by projection synthesis scheme proposed by Barnett and Pegg [7]. This enables us to map out the relative phase distribution of a condensate directly.

In the second half of this paper, we discuss how a modification to this scheme may be used to analyze the state in recent MI phase transition experiments. This is currently achieved by detecting a spatial interference pattern between the condensates and using the visibility of the fringes as a measure of the number squeezing. More recently, collapses and revivals of the macroscopic wave function have been observed and the time scales of these have been used to obtain information about the state [8]. Although both of these methods clearly demonstrate squeezing, the information they give about the quantum state is rather limited. We discuss a scheme that enables us to distinguish states much more accurately than can be achieved by these current techniques. This may allow us to probe the MI phase transition with high precision and may be of great interest to present experiments.

We begin by briefly outlining the projection synthesis scheme. A much more detailed account of this can be found in Ref. [7]. We then detail how this scheme may be implemented using current methods and discuss why it may be more feasible in the atomic rather than the optical regime.

Our starting point is the state that we wish to analyze. This takes the form of a pure state entanglement between two condensates, a and b , in adjacent lattice sites with a total of N atoms,

$$|\psi\rangle = \sum_{n=0}^N c_n |n\rangle_a |N-n\rangle_b. \quad (1)$$

To measure the relative phase distribution between a and b directly, we would need to measure $|\langle \Delta\phi | \psi \rangle|^2$, where $|\Delta\phi\rangle$ are the two-mode phase states with phase difference $\Delta\phi$ [9]. Since the entangled state (1) has a finite number of terms, we can write the probability density for it to have a relative phase $\Delta\phi$ as

$$P(\Delta\phi) = \frac{1}{2\pi} \left| \sum_{n=0}^N c_n \exp(-in\Delta\phi) \right|^2. \quad (2)$$

To measure this, we need to introduce an entangled state between modes c and d as a reference state (see Fig. 1). In close analogy with Ref. [7] this state takes the form of a reciprocal-binomial state,

$$|B\rangle = C \sum_{l=0}^N \binom{N}{l}^{-1/2} \exp\left[il\left(\Delta\phi - \frac{\pi}{2}\right)\right] |N-l\rangle_c |l\rangle_d, \quad (3)$$

where C is the normalization constant with modulus independent of $\Delta\phi$. To measure the phase probability density, Ref. [7] prescribes that we pass modes a and d through a 50:50 beam splitter which, for an atomic system, can be shown to be equivalent to Josephson coupling the traps for a quarter cycle [3]. This has been experimentally observed in a system of Raman coupled hyperfine levels of ^{87}Rb [10], and can also be achieved by allowing tunneling through the potential barrier separating condensates in an optical lattice [5,6]. We will review the full scheme shortly.

After the beam splitter, we simply record the fraction of the total trials for which N atoms were detected in D_a and no counts were detected in D_d , i.e., the number of atoms detected at lattice sites a and d , respectively, after the coupling. As discussed in Ref. [7] this is proportional to the probability that there is a relative phase $\Delta\phi$ between the condensates. This means that if we were to create an ensemble of identi-

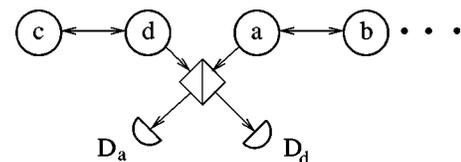


FIG. 1. The phase measurement scheme. Modes a and b form part of a lattice that we wish to analyze and c and d are reference states. We pass modes a and d through the analog of a 50:50 beam splitter and detect the final number of atoms in each output port.

cally prepared entanglements between a and b we could repeat the process for different values of $\Delta\phi$ and so plot out the whole relative phase probability density given by Eq. (2), $P(\Delta\phi)$. When this is combined with the atom number distribution, it provides all the information required to construct the whole state. This is potentially a very valuable tool, but depends on us being able to successfully create the required reference states (3).

There has been a proposal for how these reference states may be created in the optical regime [11]. However, this relies on making a particular sequence of detections which, for N photons, occurs in only about 2^{-N} of the experimental runs. This severely restricts the practicality of such a scheme. We show now how this situation is greatly improved in the atomic case since the reference states can be created deterministically. This makes the scheme much more feasible than the optical case since each experimental run will yield the desired state. For small numbers of atoms, a good approximation to the reference state (3) can be formed following the same procedure as recent number squeezing experiments in optical lattices [5,6], as we now show.

We consider the two reference state modes c and d occupying two adjacent sites in an optical lattice. This system can be described by the Hamiltonian

$$H = -J(c^\dagger d + d^\dagger c) + U(c^\dagger{}^2 c^2 + d^\dagger{}^2 d^2), \quad (4)$$

where J is the coupling strength and U is the nonlinear interaction strength. For attractive interactions ($U < 0$), which can be achieved by tuning the atomic scattering lengths [12], we adiabatically raise the potential barrier between the modes, thereby lowering the coupling between the sites, J . For adiabatic changes, the state remains in the lowest eigenstate of Eq. (4) throughout this procedure. If we stop raising the barrier when the ratio J/U reaches some optimum value, a very good approximation to Eq. (3) can be achieved for values of N up to about 6. Using a sum of squares of differences approach to compare the ground state of Eq. (4) with Eq. (3), the optimum ratios for $N=(2,3,4,5,6)$ are found to be $(J/U)_{opt} = (-0.33, -0.50, -0.67, -0.83, -1.03)$, respectively. This is very well matched by the linear relationship, $(J/U)_{opt} = -0.167N$.

For $N > 6$, this method no longer gives a good approximation to the required reference states (3). This does not matter, however, as, for all practical purposes, the method of projection synthesis is limited to small numbers anyway. The scheme prescribes that we record the fraction of trials that give a certain measurement outcome and, since this fraction decreases with increasing N , we would require an impractically large number of experimental runs to obtain satisfactory statistics for large N . Furthermore, as we shall see, this scheme is naturally suited to analyzing the MI phase transition of condensates in an optical lattice for which typical mode populations are around three atoms, which falls comfortably within our range.

The reference state creation process is completed by advancing the phase of d relative to c by $\Delta\phi - \pi/2$ using a far detuned light pulse [13]. This gives us states with the form of Eq. (3). We can now consider how well this scheme works

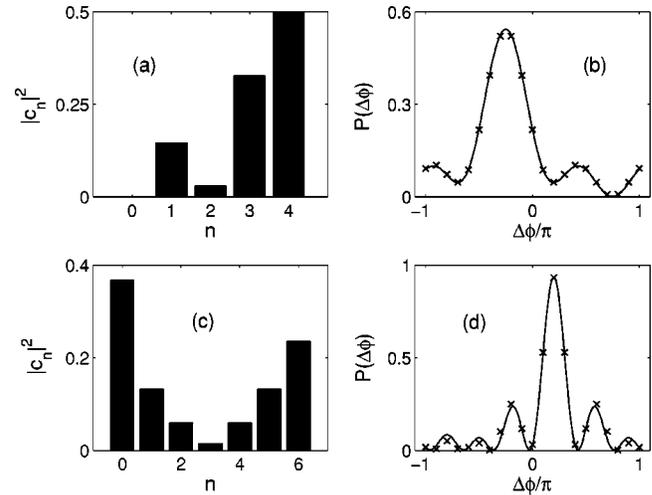


FIG. 2. The number distribution of an arbitrary state (1) is shown in (a) and the corresponding output from the phase projection scheme using the reference state with $N=4$ (crossed curve) along with the phase distribution (solid curve) are shown in (b). The corresponding results for a different arbitrary state and using the reference state with $N=6$ are shown in (c) and (d).

using these reference states. Before we do this, for the sake of clarity we will review the sequence of steps required for the whole scheme.

Our setup consists of four lattice sites formed by an optical standing wave as shown in Fig. 1, with a large initial potential barrier between a and d . The reference state is created between c and d by tuning the scattering length of the condensates in these sites to a negative value and then adiabatically decreasing J/U to some optimum value, as discussed above. A phase can then be imprinted on d relative to c to create a state of the form (3). Modes a and b are the state that we wish to analyze and so can be evolved in any way that the experimenter desires. The measurement is performed by lowering the barrier between a and d to allow Josephson coupling between these modes. This is equivalent to the action of a 50:50 beam splitter. Finally, the fraction of trials for which N atoms are detected in a and none are detected in d are recorded. A plot of these for different phases imprinted on the reference state is proportional to the relative phase density.

In Fig. 2 we compare the measurements from this scheme with the phase distribution given by Eq. (2). Figure 2(a) shows an arbitrary number distribution for state (1) and Fig. 2(b) shows the corresponding output from our scheme (crossed curve) using the reference state with $N=4$. The solid curve in Fig. 2(b) is the phase distribution calculated using Eq. (2) and shows remarkable agreement with the measurement. Fig. 2(c) and Fig. 2(d) show the corresponding results for a different arbitrary state using the reference state with $N=6$. While not quite as good as for the case $N=4$, the measured output from this scheme is still in good agreement with the phase distribution.

This demonstrates a feasible way of directly measuring the relative phase distribution between condensates. It may prove to be a very valuable tool for analyzing the state of

condensates in optical lattices. In the remainder of this paper we will focus on one particular application of this scheme which is of great current interest, namely, analyzing how close number squeezing experiments get to the perfect MI state [6]. We now discuss how this can be achieved and compare our results with the current technique of measuring a spatial interference pattern.

We consider the case of m sites in an optical lattice. We wish to detect small deviations from the Mott insulator state, $|\psi\rangle_{\text{MI}} = |3\rangle_1 |3\rangle_2 \dots |3\rangle_m$, where we have taken each mode to have three atoms, in agreement with experiments [6]. As our comparison, we will take a slightly perturbed MI state of the form

$$|\psi\rangle_{\text{pert}} = \frac{1}{\sqrt{1+\epsilon}} \left(1 + \frac{\delta}{\sqrt{6}} \sum_{\langle i,j \rangle} a_i^\dagger a_j \right) |\psi\rangle_{\text{MI}}, \quad (5)$$

where $\langle i,j \rangle$ denotes a sum over adjacent values of i and j , $\delta \ll 1$ parametrizes the perturbation, and $\epsilon = 2(m-1)\delta^2 \ll 1$ is the fraction of the total population perturbed from $|\psi\rangle_{\text{MI}}$. If, as in current experiments, we were to turn off the optical potential and allow the condensates to expand and overlap, an interference pattern would be seen. The probability of detecting the first atom at position x is given by

$$P_{\text{int}}(\theta) \approx \frac{1}{2\pi} \left[1 + \frac{4\delta}{3} \sqrt{6} \left(1 - \frac{1}{m} \right) \cos \theta \right], \quad (6)$$

where $\theta = kx$ and k is the wave number of the atomic de Broglie wave.

As the state approaches the MI state, $\delta \rightarrow 0$, the interference fringes wash out. This feature has been used in experiments as a way of demonstrating number squeezing between lattice sites [5,8]. The form of Eq. (6), however, is a best-case scenario and experiments have shown that fringes are no longer clear for relatively modest squeezing [8]. This limits the ability of such a technique to distinguish states that are close to the MI state.

We would now like to see whether an improvement can be achieved using phase projection. As discussed above, we create a two-mode reference state and project out the phase distribution of a lattice site a relative to its neighbor b . We are not interested in the distribution of atoms over the other sites and so trace over them. This leaves us with a density matrix of the form

$$\begin{aligned} \rho \approx & |3,3\rangle\langle 3,3| + \frac{\delta}{1+\epsilon} (|3,3\rangle\langle 2,4| + |2,4\rangle\langle 3,3| \\ & + |3,3\rangle\langle 4,2| + |4,2\rangle\langle 3,3|). \end{aligned} \quad (7)$$

The phase projection technique works for mixed states as well as for pure states [7], and we can calculate the phase distribution directly using

$$P(\Delta\phi) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \sum_{r,r'=0}^k \langle r, k-r | \rho | r', k-r' \rangle e^{i(r-r')\Delta\phi}, \quad (8)$$

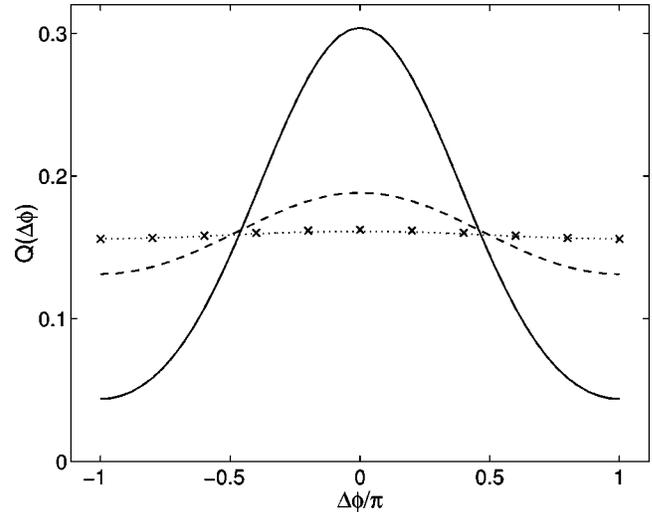


FIG. 3. The measured output from our scheme for a state (5) with $\delta=0.004$ using reference states with $J/U = -1.03$ (crossed curve), $J/U = -0.1$ (dashed curve), and $J/U = -0.02$ (solid curve). The dotted curve is the phase distribution given by Eq. (9).

which gives

$$P(\Delta\phi) \approx \frac{1}{2\pi} \left[1 + \frac{4\delta}{1+\epsilon} \cos \Delta\phi \right]. \quad (9)$$

Comparing Eq. (9) with Eq. (6), we see that, in the limit $m \rightarrow \infty$, the phase measurement scheme gives an improvement in the fringe visibility by a factor of $\sqrt{3/2}$ over measuring a spatial interference pattern. This suggests that directly measuring the relative phase distribution offers little fundamental advantage for distinguishing states close to the MI state.

Instead we would like to be able to measure something else, $Q(\Delta\phi)$, that is more sensitive to small changes in δ than the phase distribution. It turns out that there is a way this can be achieved by making only a minor modification to our phase projection scheme. In particular, when creating the reference state we consider adiabatically raising the barrier higher than the optimum value discussed above. In all other respects the measurement scheme remains unchanged. We can understand how this works by looking at the form of Eq. (7). Mode a has either 2, 3, or 4 atoms and, since we record only events for which a total of six atoms are detected, the relevant coefficients of the reference state are the ones that correspond to there being 2, 3, or 4 atoms in mode d , i.e., d_2, d_3, d_4 . In particular, the effect of the perturbed part of Eq. (7) in the measurement will be proportional to the coefficients d_2 and d_4 , and the effect of the MI part will be proportional to d_3 . This suggests that by increasing the magnitude of d_2 and d_4 relative to d_3 , we can increase the sensitivity of our measurement to the perturbation. This is precisely what is achieved in these modified reference states.

Figure 3, shows the output from this scheme operating on a state of the form of Eq. (5) with $\delta=0.004$ using different reference states with $N=6$. The crossed curve corresponds to the output using a reference state with $J/U = -1.03$ and the dotted curve is the phase distribution (9). As expected,

they agree for this value of J/U and correspond closely to the measured spatial interference pattern. We see that this distribution is very flat and it would be difficult to distinguish it experimentally from a completely flat distribution that corresponds to the pure MI state. The dashed and solid lines are the output from the same scheme but using reference states with $J/U = -0.1$ and $J/U = -0.02$, respectively. We see that the sensitivity to deviations from $|\psi\rangle_{\text{MI}}$ increases dramatically. In practice, we need only determine the ratio $Q(0)/Q(\pi)$. Any deviation from unity indicates a deviation from the MI state. For the modified reference states, even a very small deviation from the MI state results in a large deviation of this ratio from unity and should be able to be seen experimentally. For a given reference state, this ratio varies linearly with δ , and so a measurement should enable us to determine the size of the perturbation.

In conclusion, we have shown how the phase projection method of Barnett and Pegg may be applied to systems of

entangled condensates and how a good approximation to the reference state may be generated deterministically using existing experimental techniques. A comparison of the results of this scheme with the theoretical phase distribution shows excellent agreement for small numbers of atoms. This is an improvement over proposals in the optical regime and may have many uses in the analysis of condensates in optical lattices. Finally we have shown that, by slightly modifying the reference state, we may be able to distinguish the Mott insulator state from small perturbations of it with much greater resolution than by the present technique of measuring a spatial interference pattern. This may be of considerable interest for the study of the Mott insulator phase transition in current experiments.

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