Interferometry below the Standard Quantum Limit with Bose-Einstein Condensates

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We discuss a scheme for using entangled Bose-Einstein condensates to detect phase differences with a resolution better than the standard quantum limit. To date, schemes have shown that the enhancement in phase resolution gained by entangling condensates is lost when dissipation is present. Here we show how this can be overcome by using number correlated condensates, as have been produced recently in the laboratory. We also outline a scheme for measuring this phase that is not destroyed when the effects of finite detector efficiency are considered.

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Quantum limits to the noise in interferometry have long been a subject of both fundamental and practical interest. Great efforts have been made to reduce the uncertainty in interferometric measurements to below the standard quantum limit (SQL), where the precision scales inversely with the square root of the number of particles involved, N.

One way this has been achieved, in the optical regime, is by using squeezed states of light as the input to an interferometer [1,2]. Closely related to this are a number of theoretical schemes which have shown that, by introducing quantum correlations between particles, the measurement accuracy can be improved to the Heisenberg limit, where it scales as N^{-1} rather than $N^{-1/2}$ [3,4]. These schemes, which make use of macroscopic superposition states, do not, however, account for loss, and it has been shown that, when loss is included, the resolution is degraded back to the SQL [5].

Another method for achieving Heisenberg limited accuracy involves passing number correlated pairs of photons through a beam splitter [6] and has been studied when the effects of decorrelation of the input pair [7] are accounted for. The measurement in this case, however, has been shown to be extremely sensitive to any deviation from unit detector efficiencies [8], which suggests that such a scheme is unlikely to be practical. In this Letter we demonstrate how an atomic analog of this scheme also achieves Heisenberg limited accuracy and, importantly, retains sub-SQL precision even in the presence of significant losses. We also outline a procedure for measuring phase differences, which is not destroyed by imperfect detectors. This suggests that number correlated pairs may be ideal for use in high precision interferometry schemes. This is particularly timely because of recent experiments which have created such states in the laboratory [9,10].

The scheme we discuss involves manipulating condensates that are separated by an optical potential barrier and is an analog of more traditional interferometry schemes which make use of beams of particles traveling along separated paths. This new geometry has the distinct advantage of enabling us to entangle and distentangle the two paths which, as we shall see, is crucial for achieving sub-SQL measurements.

In this Letter, we are interested in the optimum phase resolution achievable for N atoms in the presence of decoherence. We compare macroscopic superposition states (Schrödinger cats) of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N\rangle|0\rangle + |0\rangle|N\rangle), \tag{1}$$

and number correlated pairs,

$$|\psi\rangle = |N/2\rangle|N/2\rangle, \tag{2}$$

with binomial states [11], which have SQL precision. To begin with, we develop the formalism for a general entanglement between two Bose-Einstein condensates.

We identify the two condensates with the annihilation operators a and b and then the general form of an entangled state, $|\psi\rangle$, with a total of N atoms may be written as

$$|\psi\rangle = \sum_{m=0}^{N} C_m|m\rangle_a|N-m\rangle_b, \tag{3}$$

where $\sum|C_m|^2 = 1$ and $\{|n\rangle\}$ are the Fock states.

The phase difference probability distribution can be evaluated by calculating the overlap between the state and the set of two-mode phase states of Barnett and Pegg differing by the angle $\Delta\theta$. This set of states constitutes the projector [12]

$$\Pi(\Delta\theta) = \sum_{l=0}^{s}|\theta_l, \theta_{(l-\Delta\theta/\epsilon)}\rangle\langle\theta_l, \theta_{(l-\Delta\theta/\epsilon)}|, \tag{4}$$

where $\epsilon = 2\pi/(s+1)$. The phase probability distribution is then

$$P(\Delta\theta, \kappa t) = \mathrm{Tr}[\rho(\kappa t)\Pi(\Delta\theta)], \quad (5)$$

where $\rho(\kappa t)$ is the density matrix that describes the evolved state at time t under the influence of dissipation at rate κ, which we take to be the same for each mode. In the case of no dissipation, this is given by $\rho(0) = |\psi\rangle\langle\psi|$. Multiplying (5) by $(s+1)/2\pi$ and taking the limit $s \to \infty$ gives the physical phase difference probability density [12,13]. This procedure leads to

$$P(\Delta\theta, \kappa t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{r,r'=0}^{n}\langle r, n-r|\rho(\kappa t)|r', n-r'\rangle \times e^{i(r-r')\Delta\theta}. \quad (6)$$

In the presence of dissipation, the density matrix, $\rho(\kappa t)$, for the system can be written as [14]

$$\rho(\kappa t) = \exp\{[\exp(2\kappa t) - 1]\hat{J}\}\exp(2\kappa t\hat{L})\rho(0), \quad (7)$$

where the superoperators $\hat{J}$ and $\hat{L}$ are defined in terms of their action on the density matrix as

$$\hat{J}\rho = a\rho a^\dagger + b\rho b^\dagger, \quad (8)$$

$$\hat{L}\rho = -\frac{1}{2}[(a^\dagger a + b^\dagger b)\rho + \rho(a^\dagger a + b^\dagger b)]. \quad (9)$$

Using (6) and (7), we can write the phase distribution of the general entangled state (3) as

$$P(\Delta\theta, \kappa t) = \frac{e^{-2N\kappa t}}{2\pi}\sum_{l,l'=0}^{N}(e^{2\kappa t} - 1)^{l+l'}\,\Omega_{l,l'}, \quad (10)$$

where

$$\Omega_{l,l'} = \left|\sum_{m=l}^{N-l'} C_m \binom{m}{l}^{1/2}\binom{N-m}{l'}^{1/2} e^{im\Delta\theta}\right|^2. \quad (11)$$

We can now use this general result to calculate the phase resolution for certain specific states. We begin by considering a binomial state, i.e., one that is formed by resonantly Raman coupling a number state $|N\rangle$ with a vacuum mode $|0\rangle$ for a quarter cycle, or alternatively by cutting a noninteracting condensate in half by adiabatically raising a potential barrier. The coefficients for this state are [15]

$$C_m = \binom{N}{m}^{1/2}. \quad (12)$$

We use this state as our benchmark since it can be formed in experiments readily and, as we shall see, its phase resolution is given by the SQL, $\Delta\theta \sim 1/\sqrt{N}$, which is what we want to surpass. Substituting (12) into (11) gives the phase distribution for the binomial state.

We define the phase resolution of the state to be the value of $\Delta\theta$ for which

$$\frac{P(\Delta\theta, \kappa t)}{P(0, \kappa t)} = \frac{1}{2}, \quad (13)$$

which corresponds to the half width at half maximum of the phase distribution.

First we consider the lossless case, which is obtained by setting $\kappa t = l = l' = 0$. If we then replace the sum in (11) with an integral, the phase distribution reduces to a Fourier decomposition of the initial number distribution. This is what we might expect since number and phase are conjugate variables. Replacing the binomial (12) with its Gaussian approximation, the calculation is straightforward and gives $\Delta\theta = \sqrt{2\ln(2)/N}$. As expected, the phase scales with N as $\Delta\theta \propto 1/\sqrt{N}$.

In Fig. 1(*a*) the solid line shows how the phase resolution, $\Delta\theta$, of this state varies when loss is introduced for $N = 50$. As more and more loss is introduced to the system, the width of the relative phase distribution increases. For large N, the phase resolution varies as

$$\Delta\theta = \sqrt{\frac{2\ln 2}{Ne^{-2\kappa t}}}. \quad (14)$$

This result is plotted as the crossed curve in Fig. 1(*a*) and shows good agreement with the full calculation. The form of (14) is simply the resolution given by a binomial state with a reduced number of atoms due to the loss and suggests that a binomial state is not changed by loss. Another way to think of this is that the binomial state is very robust [16].

With the benchmark set, we now try to improve upon it. We start by considering the phase resolution of a cat state.

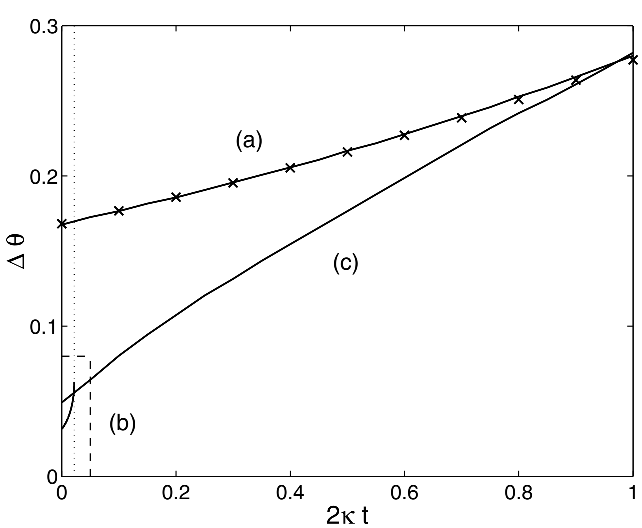

FIG. 1. Variation of the relative phase resolution with $2\kappa t$ for (*a*) a binomial state (solid curve) and the approximation given by (14) (crossed curve), (*b*) a cat state, and (*c*) a number correlated state for $N = 50$. The dotted line indicates the point at which the cat state has been destroyed and no longer contains any phase information. The dashed region is enlarged in Fig. 2.

We choose such a state since it is highly entangled and there have been proposals for how it may be able to be used to give Heisenberg limited resolution in interferometry schemes [3]. The coefficients for this state are

$$C_m = (\delta_{m,0} + \delta_{m,N})/\sqrt{2}, \quad (15)$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$.

Substituting into (11), the relative phase distribution is

$$P(\Delta\theta, t) = \frac{1}{2\pi}[1 + e^{-2N\kappa t}\cos(N\Delta\theta)]. \quad (16)$$

A simple calculation yields the phase resolution, which is given by

$$\cos(N\Delta\theta) = \frac{1}{2}(1 - e^{2N\kappa t}). \quad (17)$$

In the limit of no loss, $\kappa t = 0$, the solution is $\Delta\theta = \pi/(2N)$. The phase resolution of this state scales as $\Delta\theta \sim 1/N$ and potentially allows for a considerable improvement over the binomial case which, as we have seen, scales as $1/\sqrt{N}$. This fact has led to proposals for the use of cat states in clock schemes [4].

If there is loss, however, Eq. (17) has no solution for $N\kappa t > \ln(3)/2 \approx 0.5$. This means that the phase information is completely wiped out if on average more than half of an atom is lost from each mode, i.e., if at least one atom is lost from the system. This is what we would expect since a macroscopic superposition state of this sort is completely destroyed by the loss of a single atom. We can infer from this that the cat state has a lifetime of $t \approx 1/(2N\kappa)$.

In Fig. 1(*b*) we have plotted how the solution, $\Delta\theta$, of (17) varies with $2\kappa t$ for $N = 50$. An enlargement is shown in Fig. 2. For no loss, the resolution is significantly better than for a binomial state with the same number of atoms. However, it rapidly worsens as κt is increased and, as predicted, the relative phase is undefined for $2\kappa t > \ln(3)/50 \approx 0.022$.

Although cat states have excellent phase resolution properties, they are extremely fragile and have a fleeting existence in the presence of loss. Huelga *et al.* showed that when it comes to usefulness for a frequency standard these two effects exactly cancel [5]. A cat state does not give better phase resolution than a standard quantum limited state since, although cat states are more sharply defined in phase space, they can be evolved only for a much shorter time.

This standard quantum limit, however, is by no means fundamental, and it has been shown that it can be surpassed by a factor of $1/\sqrt{e}$ by making use of partially entangled states with a high degree of symmetry [5]. We now show how sub-SQL resolution can be achieved by making use of number correlated condensates, which have the important advantage of being able to be created experimentally [9,10].

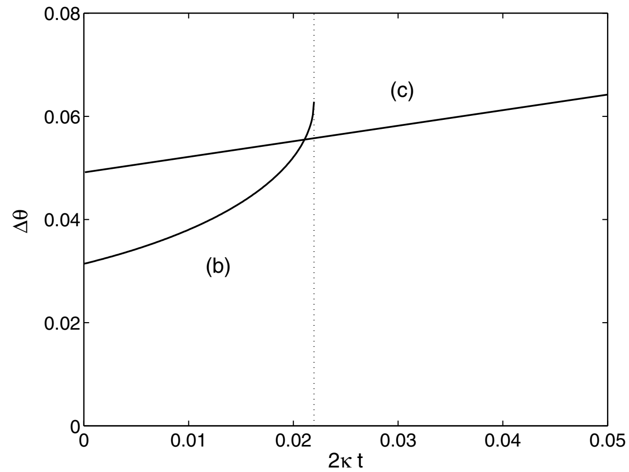

FIG. 2. An enlargement of the dashed region in Fig. 1.

The phase resolution of number correlated pairs of photons of the form of (2) are known to scale as $1/N$ when passed through a beam splitter [6]. An analogous result has been shown for resonantly Raman coupled (or Josephson coupled) pairs of atoms [17]. Intuitively, we might feel that these states contain more information than cat states since it has been shown that cat states can be formed from number correlated pairs when there is a reduction of coherence or information due to loss [18]. We now investigate this more formally and show how this allows us to surpass the SQL.

Raman coupling the number correlated state $|\psi\rangle = |n\rangle|n\rangle$, with a total number of $N = 2n$ atoms, for a quarter cycle yields the state [6]

$$|\psi\rangle = \frac{1}{2^n}\sum_{m=0}^{n}\frac{\sqrt{(2m)!(2n-2m)!}}{m!(n-m)!}|2m\rangle|2(n-m)\rangle. \quad (18)$$

By substituting the coefficients into (11) and taking the lossless limit, $\kappa t = l = l' = 0$, the phase resolution for this state can be shown to vary as $\Delta\theta \approx 1.4/n = 2.8/N$ for large N. This has the same number scaling as the cat state ($\Delta\theta = \pi/2N$) but is a factor of about 1.8 worse. We can conclude that in a purely lossless situation, cat states would be the best states to use in interferometry.

A completely lossless situation, however, is not realistic and we need to compare the two states in the presence of dissipation. As before, we can find $\Delta\theta$ by solving (13) where $P(\Delta\theta, \kappa t)$ is given by (10). In Fig. 1(*c*) we plot $\Delta\theta$ for different values of $2\kappa t$ for $N = 50$ using the coefficients given by (18). We can compare this with Figs. 1(*a*) and 1(*b*) which show the corresponding relationships for a binomial state and a cat state, respectively. In the lossless case, the phase resolution of this state scales with N in the same way as for the cat state. Their behavior differs markedly, however, when loss is present.

Although the loss of a single atom destroys all the phase information in a cat state, we see in Fig. 1(*c*) that for the number correlated state the loss of a single atom only slightly degrades the phase resolution. As the loss is increased, the phase resolution is smoothly degraded but a well defined relative phase still exists between the modes. This is true even for significant losses. In Fig. 1 the phase resolution is plotted for losses of over 60% of the atoms from the original system. An analysis for different numbers of atoms shows that the number scaling of the phase resolution, which is $1/N$ at $\kappa t = 0$, tends towards $1/\sqrt{N}$ as κt increases. Huelga *et al.* showed that for uncorrelated particles (corresponding to the binomial state) the optimum phase resolution is achieved after an evolution time $t = 1/(2\kappa)$ [5]. Over this range, we see that the number correlated state has better phase resolution than the binomial state and so must allow for improved measurement precision.

This result is very encouraging as our required starting state (number correlated modes) has been observed in chains of coupled condensates [9] and perfect number correlation, via a Mott insulator transition, has been experimentally demonstrated in a three-dimensional lattice [10]. A further analysis shows that the modes do not need to be perfectly number correlated to achieve sub-SQL precision.

An experimental implementation of this scheme would require a readout process. Current readout schemes for photons involve passing the state through a second beam splitter and detecting the number difference between the two output ports, $J_z = (a^\dagger a - b^\dagger b)/2$ [6,7]. Kim *et al.* have shown that $\langle J_z \rangle$ for such a technique is always zero and that the relative phase difference is encoded in the correlations between the output ports $\langle J_z^2 \rangle$ [7]. Furthermore, they have shown that, for an N-particle state, the detectors must have an efficiency better than $1 - 1/N$ to detect this signal. This renders such a technique impractical for large N.

An alternative measurement scheme can be applied to the atomic case. After passing the state through the second beam splitter (i.e., resonant Josephson coupling), we pass it back through the Mott transition. This can be achieved by adiabatically lowering the potential barrier between the wells [9,10], which disentangles the particles. Numerical simulations we have done show that, for a relative phase difference ϕ, the mean number difference between the two traps at the end of this process scales as

$$\langle J_z \rangle \propto \sqrt{N} \sin(N\phi), \tag{19}$$

and the variance scales as $(\Delta J_z)^2 \propto N$. This means that the phase can be resolved to an uncertainty [19]

$$\Delta\phi = \frac{\Delta J_z}{\partial J_z / \partial \phi} \propto \frac{1}{N}; \tag{20}$$

i.e., it is Heisenberg limited. The key point, however, is that the relative phase has now been encoded on the mean population difference between the traps, rather than on the noise. This means that any deviation from unit detector efficiency only linearly degrades the signal (19). In other words, the phase difference should still be able to be readout even for detectors with moderate efficiencies. The full details of this measurement process will be presented elsewhere.

Our results suggest that highly number-squeezed Bose-Einstein condensates may be ideal candidates to be used in interferometry schemes since they combine Heisenberg limited precision with robustness to loss. Furthermore, by passing the state back through the Mott transition, problems associated with finite detector efficiencies can be overcome. This technique could have applications in precision measurements of such things as frequencies in clock schemes or forces.

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