Entanglement concentration in Bose-Einstein condensates

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We propose a scheme for demonstrating entanglement swapping (i.e., teleportation of entanglement) using trapped Bose-Einstein condensates. This is accomplished by detection of the total number of atoms leaking out of two adjacent traps. We describe how this scheme may be used to concentrate entanglement shared between two parties in the form of entangled condensates.

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Entanglement shared between distant parties is an important resource in various communication protocols [1,2]. Since entanglement cannot be created by local operations on separate systems, entangled pairs of systems need to be created at a source and then distributed to distant parties. It is, therefore, important to be able to distribute entanglement faithfully.

One scheme that aids in this distribution is entanglement concentration. This converts a large number of less entangled pairs into a smaller number of more entangled pairs using only local operations [3]. The concentrated pairs can then be used to perform the communication protocol faithfully. One such protocol, quantum teleportation [2], exploits priorly shared entanglement to transfer an unknown (or even partially unknown [4]) quantum state to a distant location. Teleporting one particle of an entangled pair results in the exchange of entangled partners across a distance, a process known as ''entanglement swapping'' [2,5]. Two of the particles that become entangled have not directly interacted at any stage, and, therefore, the swapping can be used as a means of entanglement distribution. Both teleportation and entanglement swapping have been experimentally demonstrated [6,7], in the domain of microscopic systems. Demonstrations of such schemes with macroscopic objects would be of fundamental interest.

An increasingly controllable macroscopic quantum system is a Bose-Einstein condensed state of a trapped dilute gas [8]. Bose condensates have already been used to demonstrate quantum features, such as superposition, in the macroscopic domain [9]. More recently, there have also been suggestions for creating specific types of entangled atomic beams, and multiparticle entangled states from Bose condensates [10]. In this paper, we propose a method for achieving entanglement swapping, using techniques that are experimentally feasible. This opens up the possibility of entanglement-aided communications using Bose condensates. We also show that a theoretical proposal for entanglement concentration by entanglement swapping, [11], may be implemented, using exactly the same measurements.

We begin by presenting a general entanglement swapping protocol for Bose condensates. We consider four condensates labeled by the modes a,b,c, and d (see Fig. 1). Modes a and b are entangled with a total number of n atoms

$$|\psi_{ab}\rangle_1=\sum_{p=0}^{n} c_p|n-p\rangle_a|p\rangle_b, \tag{1}$$

where $\Sigma_{p=0}^{n}|c_p|^2=1$. The second two modes, c and d, are also entangled with a total number of n atoms

$$|\psi_{cd}\rangle_1=\sum_{q=0}^{n} d_q|q\rangle_c|n-q\rangle_d \tag{2}$$

with $\Sigma_{q=0}^{n}|d_q|^2=1$. We will discuss specific ways of entangling these condensates later. The total state of the system after this first part of the scheme is, $|\psi\rangle_1=|\psi_{ab}\rangle_1\otimes|\psi_{cd}\rangle_1$, containing a total of $2n$ atoms.

In the second part of the scheme, we allow atoms to leak out of b (now entangled with a) and c (now entangled with d). We completely drain the two traps and detect all the atoms in both of them in such a way that we do not know which trap a detected atom has come from. We need to be able to measure this number accurately which, for ground-state condensates, constrains this method to small numbers of atoms. This may still be of interest in the context of optical lattices where average mode populations can typically be only a few atoms [12]. Alternatively, it has been suggested that accurate counting of much larger samples of atoms can be achieved by using condensates of metastable helium, ^{4}He* [13]. As we are not concerned with preserving modes b and c in this process, the details of this step can be very general. Any method that measures a spatial or temporal interference pattern should work [9,14–16]. We can write this step as the operation

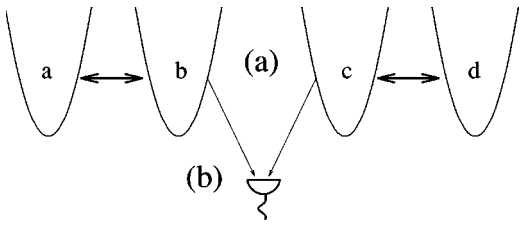

FIG. 1. The entanglement swapping scheme. (a) Modes a and b are entangled as are modes c and d. (b) An interference pattern is then detected between modes b and c, completely draining both traps.

$$\mathcal{O}(p,q)|\psi\rangle_1 = \delta_{r,(p+q)}\left[\frac{1}{\sqrt{2}}(\hat{b}+\hat{c})\right]^{p+q} e^{i\xi q}|\psi\rangle_1, \quad (3)$$

where $\hat{b}$ and $\hat{c}$ are annihilation operators for modes b and c, respectively, $\delta_{r,(p+q)}$ is the discrete delta function, and r is the outcome of the measurement of the total number of atoms. The phase ξ varies stochastically from trial to trial [14] since it has contributions from the position or time of each detection, and these detections are random events. The value of this phase, however, is not important to our scheme. In fact, detecting an interference pattern not only creates a phase, but measures its value, [17], so we could easily correct for it at the end. For this reason and without loss of generality we set $\xi=0$.

Considering just modes b and c, we get

$$\mathcal{O}(p,q)|p\rangle_b|q\rangle_c = \frac{(p+q)!}{\sqrt{p!q!}}\delta_{r,(p+q)}|0\rangle_b|0\rangle_c. \quad (4)$$

Modes b and c have been completely drained and so we will omit these modes from the expression for the total state. Making the replacements $y=n-p$ and $k=2n-r$, the final state of modes a and d can be written as

$$|\psi\rangle \propto \sum_{y=y_1}^{y_2} \binom{2n-k}{n-y}^{1/2} c_{n-y} d_{n-k+y}|y\rangle_a|k-y\rangle_d, \quad (5)$$

where $y_1=\max(0,k-n)$ and $y_2=\min(n,k)$. Modes a and d are now entangled, though they have never directly interacted. This completes the entanglement swapping protocol. The physical basis of the swapping is the lack of knowledge of the source of the detected atoms. This is also the principle on which a recent proposal for entangling atoms in distant cavities is based [18].

The fact that entanglement swapping is possible leads us to investigate whether it could be used for entanglement concentration, as indicated in Ref. [11]. This depends on the choice of initial states (1) and (2). We investigate the circumstances under which the entanglement of the final state (5) may be greater than that of the original state (1). In this case, mode a may be taken to belong to Alice and modes b, c, and d to Bob. The detection, a local action on Bob's side, then serves to increase the entanglement between Alice and Bob. Entanglement concentration is thus demonstrated, in principle, between Alice and Bob.

To compare the entanglement of the initial and final states (2) and (5), we will use a standard measure of entanglement for pure states, namely, the Von Neumann entropy of one subsystem [3]. For state (5), the entanglement is then given by

$$E = -\mathrm{Tr}\{\rho_a \ln \rho_a\} = -\mathrm{Tr}\{\rho_d \ln \rho_d\}, \quad (6)$$

where $\rho_a=\mathrm{Tr}_d\{|\psi\rangle\langle\psi|\}$ and $\rho_d=\mathrm{Tr}_a\{|\psi\rangle\langle\psi|\}$, respectively, denote the reduced density matrices for modes a and d.

The range of possible initial states is limited by the specific practical methods of entangling the initial pairs of condensates. A number of theoretical schemes could be proposed for which the entanglements and measurements are not practical. However, all the operations in our scheme could be performed in the laboratory. We need to make measurements that entangle modes a and b without destroying the resulting state. To this end, we consider coherently coupling a component out from each mode and making measurements on these components. This has already been experimentally demonstrated for condensates of sodium [19] and rubidium [20]. Robert *et al.* have discussed how a similar result may be achieved with ^{4}He* by transferring population to the untrapped $m=0$ magnetic sublevel using a laser induced Raman transition [13]. The two outcoupled components could then be allowed to fall onto an array of detectors and an interference pattern measured. For simplicity and convenience, we consider each mode initially to be in a number state with n atoms with identical trap frequencies ω and that we detect n atoms (i.e., half of them) in the entanglement process. Such a procedure gives the unnormalized coefficients in Eq. (1) as

$$c_p = \binom{n}{p}^{3/2} e^{-i\zeta p}. \quad (7)$$

As above, ζ can be set to zero without loss of generality.

We would now like to find an experimentally feasible form of the entanglement between modes c and d (held by Bob) that can give rise to purification of the entanglement between Alice's mode a and Bob's mode d.

If we consider the form of the final state (5) with $r=k=n$ (i.e., the final state has the same number of atoms as the initial state) and substitute the coefficients from Eq. (7), we find

$$|\psi\rangle \propto \sum_{y=0}^{n} \binom{n}{y}^2 d_y|y\rangle_a|n-y\rangle_d. \quad (8)$$

The form of this, with a combinatorial factor that is peaked in the middle, suggests that we require a state (2) with a dipped number distribution d_y. This would broaden the peaked distribution of Eq. (8) towards a flat maximally entangled final state, thereby realizing concentration of entanglement. It is not clear how this could be achieved perfectly in the laboratory, however, we show how an approximation may be able to be realized.

We start by entangling c and d just as we produced the entanglement between a and b. It is assumed that this process takes place much faster than the evolution due to the nonlinear interactions. The system is then allowed to evolve naturally due to the interactions between atoms. We take the modes to have the same interaction strength U and assume that there is no interaction between the two modes. Roberts *et al.* [21] have shown how the scattering length may be tuned for ^{85}Rb, and a similar scheme may be useful in a practical implementation of this step. The system is allowed to evolve for time $t=\pi/4U$. The operator for this step is

$$U_{\mathrm{evolve}} = \exp\left[i\frac{\pi}{4}(\hat{c}^{\dagger 2}\hat{c}^2 + \hat{d}^{\dagger 2}\hat{d}^2)\right], \quad (9)$$

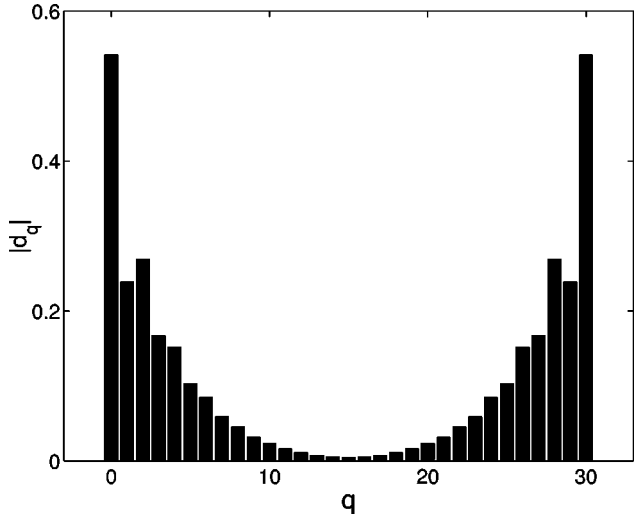

FIG. 2. Plot of the coefficients $|d_q|$ against q, for $n=30$ and $\phi=0.8$.

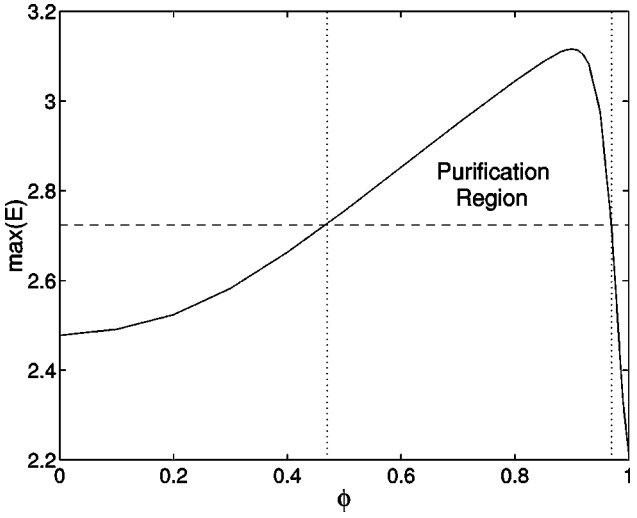

FIG. 3. The maximum entropy over all possible measurement outcomes r of the final state (5) as a function of ϕ. The dashed line is the entropy of the initial state (1).

and is equivalent to the ''one-axis twisting'' described by Kitagawa and Ueda [22]. Next the phase of one condensate is advanced by $\phi\pi/2$ by applying an appropriate light pulse [23]. The state after this step is

$$|\psi_{cd}\rangle_1 \propto \sum_{q=0}^{n} \binom{n}{q}^{3/2} e^{i\pi q(q-n-\phi)/2} |q\rangle_c |n-q\rangle_d . \qquad (10)$$

Finally, we resonantly couple modes c and d (by tunneling through the barrier between the traps) for time $t=\pi/4\Gamma$, where Γ is the strength of the coupling [24], (this Hamiltonian is identical to that generated by Raman coupling [25]). The resulting unitary operator is

$$U_{\text{couple}} = \exp\left[i \frac{\pi}{4} (\hat{c}\hat{d}^\dagger + \hat{c}^\dagger \hat{d}) \right]. \qquad (11)$$

Operating on Eq. (10) gives us the form of Eq. (2). After some algebra, the unnormalized coefficients are given by

$$d_q = \sqrt{q!(n-q)!} \quad \sum_{s=0}^{n} \sum_{t=t_1}^{t_2} \binom{n}{s}^2 \binom{s}{t} \binom{n-s}{q-t} \times e^{i\pi(q-2t)/2} e^{i\pi s(s-n-\phi)/2}, \qquad (12)$$

where $t_1 = \max(0, q+s-n)$ and $t_2 = \min(q,s)$. A plot of these coefficients for $n=30$ and $\phi=0.8$ is shown in Fig. 2. We see that this has the general dipped form of the number distribution that we require.

As described in the entanglement swapping scheme above, once the two entangled pairs have been created, we can measure an interference pattern between modes b and c, detecting the total number of atoms in both. An expression for the final state is then given by substituting the coefficients given by Eqs. (7) and (12) into Eq. (5). The final state depends on three parameters: the total number of atoms detected r, the phase ϕ introduced in creating Eq. (12), and the number of atoms in each entangled pair n. For simplicity, we will not investigate the n dependence but will set $n=30$ throughout the remainder of this paper. We are not so much interested in presenting the best case of entanglement purification, as demonstrating that it is possible. We will, however, examine how the entanglement of Eq. (5) varies with r and ϕ.

We begin by investigating the optimum value for ϕ. In Fig. 3, the maximum value of the entanglement of Eq. (5) that can be achieved for any measurement result r is plotted as a function of ϕ. For comparison, the dashed line is the entanglement of the initial state (1). We see immediately that the entanglement is purified over a wide range of values of ϕ: $\phi \in [0.47, 0.97]$. This is an important result and demonstrates a means of purifying the entanglement of multipar-

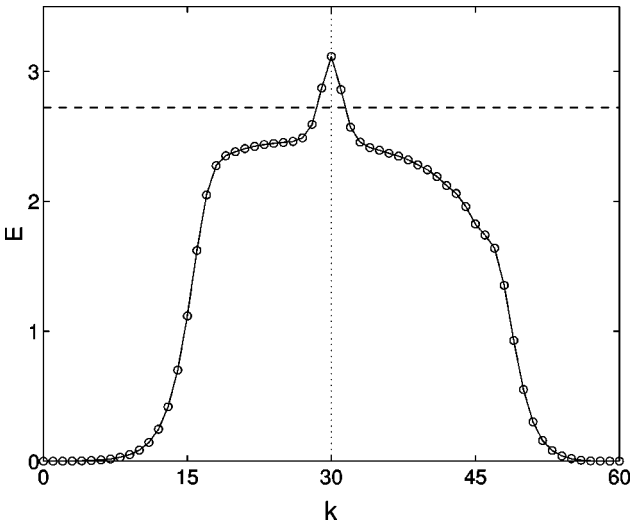

FIG. 4. Entropy of the final state (5) as a function of the total number of atoms in the state k. The dashed line is the entropy of the initial state (1). The entanglement is purified over the range $k \in [n-1, n+1]$, with a maximum for $k=n=30$.

ticle systems. For $n=30$ the best entanglement of the final state is achieved when $\phi=0.9$. We now set ϕ to this optimum value and investigate how the entanglement of the final state depends on the outcome of the measurement of the total number of atoms in modes b and c. In particular, we would like to study the range of outcomes that give rise to entanglement purification. In Fig. 4, we plot E against the total number of atoms remaining in the final state, $k=2n-r$. We see that the final state is purified if $k\in\{n-1,n,n+1\}$, i.e., the measurement outcome lies in the range $r\in\{n-1,n,n+1\}$. If any other measurement result is obtained, the trial has failed to purify the entanglement and we would need to try again.

Of course, one would need to know accurately the number of atoms in each entangled pair. This may be able to be achieved by making use of coupled condensates in an optical lattice to create an array of number correlated condensates, as has been experimentally demonstrated [12]. Taking any five of these lattice sites, we could accurately measure the number of atoms at one site, which would leave four sites each containing the same known number of atoms. This is precisely the starting state we require.

We see from Fig. 4 that if we were to average over all measurement outcomes, the entanglement would be degraded. However, conditional on a small range of measurement outcomes, it is possible to concentrate the entanglement. This is precisely the idea of entanglement concentration: from a large number of entangled pairs, we can distill a set with enhanced entanglement.

We have shown that it is possible to concentrate the entanglement of Bose condensates and have demonstrated a feasible scheme for carrying out this protocol. We have been concerned not so much with demonstrating the best possible scheme or one with any particular practical advantages as showing that it is possible. The ability to manipulate and, in particular, purify the entanglement of macroscopic objects is of fundamental importance. Techniques for manipulating entanglement are progressing rapidly and improved purification schemes should follow. The ability to concentrate the entanglement of Bose-Einstein condensates suggests that they may become a valuable tool in quantum information experiments.

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