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Transitivity of the relative localization of particles

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Abstract. We discuss the emergence of 'classical' relative positions between delocalized quantum particles when light is scattered off them. These well-defined separations are due to entanglements between the particles and can occur even though the absolute position of each particle remains undefined. This suggests that the natural spatial coordinates for such a system are those of relative position. We check that these relative localizations provide a consistent coordinate space by demonstrating their transitivity and we discuss how they may emerge in a multiparticle system.

1. Introduction

A key issue in understanding the boundary between quantum and classical physics is how objects localize in position space. One description of this process that has received a lot of attention is the theory of decoherence in which a system is coupled to an environment with many degrees of freedom, causing the decay of its macroscopic coherences [1–4]. This theory, however, is often difficult to apply because of the complex nature of the interaction with the environment. More recently, there has been a simpler proposal for how well-defined 'classical' relative positions can emerge between quantum particles due to entanglement [5]. In this scheme, photons are scattered off two delocalized particles and detected in the far-field. This process entangles the two particles in momentum space and a feedback mechanism ensures that robust relative positions are generated between them even though the absolute position of each particle remains undefined. This result is interesting not only because it demonstrates how robust classical positions can emerge from quantum superpositions, but also because it suggests that the natural spatial framework for quantum mechanics is relative position.

An important question that arises from this scheme is whether the relative positions that are generated provide a consistent coordinate space. In this paper, we address this issue by investigating whether the relative positions that emerge are transitive. In other words, if we were to measure a relative position between two particles, 1 and 2, and then do the same between particle 2 and a third particle, 3, would we be able to predict the result of a measurement of the separation of particles 1 and 3? A similar question has been addressed in the context of the quantum phase of atomic Bose–Einstein condensates [6].

It may seem trivial that transitivity should hold, but this is only obvious in the case of well-defined classical positions. In the localization scheme presented here, the positions of all the particles are undefined and remain so throughout

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the measurement process. Instead, we need to investigate whether the relative positions due to the build-up of entanglement between the particles are transitive. It is not clear that this must be the case.

In this paper, we shall first develop a formalism for the scattering process in order to check the transitivity result for single scattering events. We shall then generalize this scheme by iterating it to simulate multiple scatterings. Finally, we shall consider how a framework of relative positions can emerge in a multiparticle system. Before we do any of this, however, it is useful to review the localization scheme for two particles.

2. Localization scheme

We consider two particles that are initially in a superposition of different locations, x_1 and x_2, respectively. This state can be written as

$$|\psi_0\rangle = \sum_{x_1, x_2} b(x_1)c(x_2)|x_1, x_2\rangle. \tag{1}$$

We take the coefficients $b(x_1)$ and $c(x_2)$ to be normalized flat distributions, i.e. we do not assume any initial localization for particles 1 and 2, and $\sum |b(x_1)|^2 = 1$ and $\sum |c(x_2)|^2 = 1$. Next, we illuminate these delocalized particles with optical plane waves with wavenumber k, as depicted in figure 1. The photons scatter at some angle θ and are detected in the far-field. For simplicity, throughout this paper we will consider the relative position between the particles only in the x direction (as shown in figure 1). However, all the results can readily be extended to three dimensions.

For a photon scattered at angle θ the momentum kick to the particles in the x direction is $\Delta k = k \sin\theta$. Since we cannot know which particle the photon scattered from, the state is put into a superposition of a momentum kick to each particle.

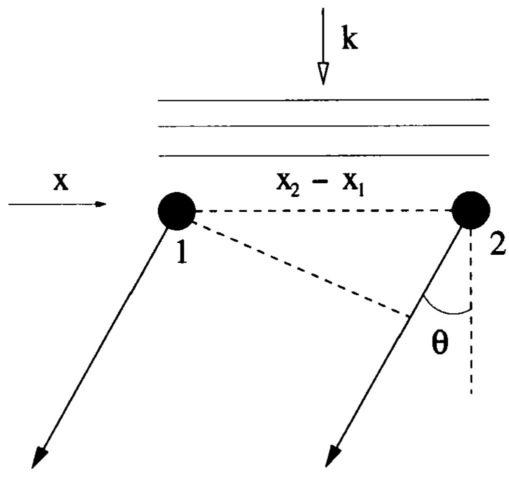

Figure 1. Optical plane waves are incident on a pair of particles. Photons are scattered at some angle θ from particles 1 and 2. The scattered photons are measured at detectors in the far-field and the scattering angle is recorded.

After a single scattering event, the state just before the photon is detected is therefore [5]

$$|\Psi_0\rangle = \frac{1}{2(2\pi)^{1/2}} \sum_{x_1, x_2} b(x_1)c(x_2) \times \left[\int_0^{2\pi} d\theta \left(\exp(ikx_1 \sin\theta) + \exp(ikx_2 \sin\theta) \right) |\theta\rangle + A|0\rangle \right] |x_1, x_2\rangle,$$

where the square brackets contain the photon state and $|\theta\rangle$ represents a photon scattered at angle θ. Throughout this paper, we will use lower case $|\psi\rangle$ to represent the state of the particles and upper case $|\Psi\rangle$ to represent the entangled system of the particles and the photon. This is the same notation as used in [5]. The term proportional to

$$A = \left[\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \left[\frac{1}{2} k(x_2 - x_1) \sin\theta' \right] d\theta' \right]^{1/2}, \tag{3}$$

represents a non-scattering event and is necessary since the total scattering rate depends on the relative separation of the particles. The probability for a photon to be detected at angle θ can be calculated directly from (2) and is given by

$$P(\theta \neq 0) = \langle \Psi_0 | \hat{a}_\theta^\dagger \hat{a}_\theta | \Psi_0 \rangle = \frac{1}{2\pi} \sum_{x_1, x_2} |b(x_1)|^2 |c(x_2)|^2 \cos^2 \left[\frac{1}{2} k(x_1 - x_2) \sin\theta \right], \tag{4}$$

$$P(\theta = 0) = \frac{1}{2\pi} \sum_{x_1, x_2} |b(x_1)|^2 |c(x_2)|^2 \int_0^{2\pi} \sin^2 \left[\frac{1}{2} k(x_1 - x_2) \sin\theta' \right] d\theta', \tag{5}$$

where $\hat{a}_\theta$ is the annihilation operator for a photon scattered at angle θ. This probability distribution is properly normalized, $\int_0^{2\pi} P(\theta)\, d\theta = 1$.

We see from (4) and (5) that, if $b(x_1)$ and $c(x_2)$ are completely flat distributions, the probability distribution for the angle of detection does not depend on the absolute positions of the particles, x_1 and x_2, but only on the relative position, $x_{12} = x_1 - x_2$. It is convenient, therefore, to rewrite the state in terms of this variable. The initial state becomes

$$|\psi_0\rangle = \sum_{x_2} c(x_2)|x_2\rangle \otimes \sum_{x_{12}} d(x_{12})|x_{12}\rangle, \tag{6}$$

where $d(x_{12})$ is taken to be a normalized uniform distribution, i.e. the relative position between 1 and 2 has a large uncertainty. For this new state, the probability of detecting a photon at angle $\theta \neq 0$ follows from (4) and (6),

$$P(\theta) = \frac{1}{2\pi} \sum_{x_{12}} |d(x_{12})|^2 \cos^2 \left(\frac{1}{2} k x_{12} \sin\theta \right), \tag{7}$$

where we have used $\sum |c(x_2)|^2 = 1$. We can simulate a scattering event by randomly selecting an angle of detection, $\theta = \Theta_{12}$, from this distribution (7). After this scattered photon is detected, the new state, $|\psi_1\rangle$, is

$$\begin{aligned} |\psi_1\rangle &= \hat{a}_{\Theta_{12}}|\Psi_0\rangle \\ &\propto \sum_{x_2} c(x_2) \exp(\mathrm{i}kx_2 \sin\Theta_{12})|x_2\rangle \\ &\quad \otimes \sum_{x_{12}} d(x_{12})(1 + \exp(\mathrm{i}kx_{12}\sin\Theta_{12}))|x_{12}\rangle. \end{aligned} \tag{8}$$

We can calculate the relative localization between particles 1 and 2 directly from this state by using the density matrix, $\rho = |\psi_1\rangle\langle\psi_1|$. The reduced density matrix for x_{12} is found by tracing ρ over x_2, i.e. $\rho_{12} = \mathrm{Tr}_2\{\rho\}$. The probability distribution for particles 1 and 2 to have a separation of x_{12} is then

$$P(x_{12}) = \langle x_{12}|\rho_{12}|x_{12}\rangle \propto |d(x_{12})|^2 \cos^2(\alpha x_{12}), \tag{9}$$

where $\alpha \equiv k \sin\Theta_{12}/2$. Since the $d(x_{12})$ distribution is flat, this probability density has maxima at

$$x_{12} = X_{12} = \frac{\pi n}{\alpha}, \tag{10}$$

where n is an integer. This means that by detecting a photon scattered from two delocalized particles, a relative localization is induced. This process can be iterated to model multiple scattering events. In this case, subsequent detection events are conditioned on earlier results, which feedback into the system and successively narrow the position probability density [5–7].

It is instructive to consider the position of particle 2 after the scattered photon has been detected. We can calculate this by tracing the total density matrix $\rho = |\psi_1\rangle\langle\psi_1|$ over x_{12} to give $\rho_2 = \mathrm{Tr}_{12}\{\rho\}$, and the probability density for particle 2 to be located at x_2 is then

$$P(x_2) = \langle x_2|\rho_2|x_2\rangle \propto |c(x_2)|^2. \tag{11}$$

This is a uniform distribution and remains so after any number of scattering events, which means that the absolute position of particle 2 (and hence also of particle 1) remains ill-defined. As we have seen above, however, this does not preclude well-defined relative positions from developing between the particles.

3. Transitivity

Now that we have outlined the formalism for the localization scheme, we wish to investigate whether the relative positions generated by this method are transitive. We consider a situation similar to that depicted in figure 1, but now with the addition of a third delocalized particle, 3. In this case, we can write the initial state in position space as

$$|\psi\rangle = \sum_{x_2} c(x_2)|x_2\rangle \otimes \sum_{x_{12}} e(x_{12})|x_{12}\rangle \otimes \sum_{x_{23}} f(x_{23})|x_{23}\rangle, \tag{12}$$

where $x_{23} = x_2 - x_3$ and $c(x_2)$, $e(x_{12})$ and $f(x_{23})$ are all normalized flat distributions, i.e. we do not assume an initial relative position between any pair of particles.

As above, the procedure starts by detecting a single photon scattered off particles 1 and 2 at an angle $\theta = \Theta_{12}$. Next we detect a single photon scattered off particles 2 and 3 at an angle $\theta = \Theta_{23}$. By a straightforward extension of the two particle analysis above, the state after these two scattering events and detections can be written as

$$|\psi\rangle \propto \sum_{x_2} c(x_2) \exp[2i(\alpha+\beta)x_2]|x_2\rangle \otimes \sum_{x_{12}} e(x_{12})(1+\exp(2i\alpha x_{12}))|x_{12}\rangle \otimes \sum_{x_{23}} f(x_{23})(1+\exp(-2i\beta x_{23}))|x_{23}\rangle, \tag{13}$$

where $\alpha \equiv (k \sin\Theta_{12})/2$ as before and $\beta \equiv (k\sin\Theta_{23})/2$. By writing the state in this form, we see that the absolute position of particle 2, x_2, is independent of the relative positions between the particles, x_{12} and x_{23}. The absolute position of particle 2 (or any other particle) remains undefined throughout this measurement process. For this reason, we can simplify the state in the remainder of this analysis by neglecting $|x_2\rangle$. We can also ignore the coefficients e and f since they are uniform. With these simplifications, the state has the form

$$|\psi\rangle \propto \sum_{x_{12}} [1+\exp(2i\alpha x_{12})]|x_{12}\rangle \otimes \sum_{x_{23}} [1+\exp(-2i\beta x_{23})]|x_{23}\rangle. \tag{14}$$

The reduced density matrix for x_{23} is found by tracing ρ over x_{12}, i.e. $\rho_{23} = \mathrm{Tr}_{12}\{\rho\}$, and the probability distribution for particles 2 and 3 to have a separation of x_{23} is then

$$P(x_{23}) = \langle x_{23}|\rho_{23}|x_{23}\rangle \propto \cos^2(\beta x_{23}). \tag{15}$$

The most probable separations for this distribution are

$$x_{23} = X_{23} = \pi m/\beta, \tag{16}$$

where m is an integer. In a similar manner, we can calculate the probability distribution for the separation between particles 1 and 2,

$$P(x_{12}) = \langle x_{12}|\rho_{12}|x_{12}\rangle \propto \cos^2(\alpha x_{12}). \tag{17}$$

In this case, the most probable separations are $x_{12} = X_{12} = \pi n/\alpha$, where n is an integer. We notice from (10) that these are the same values as those induced by the first scattering event. In other words, the second scattering event does not disrupt the pre-existing separation between 1 and 2. This feature is crucial for ensuring transitivity between the relative positions.

Our final task to demonstrate transitivity is to show that the relative localizations obey the relationship

$$X_{13} = X_{12} + X_{23}. \tag{18}$$

To confirm this, we need to calculate X_{13}. We can do this by rewriting the state (14) in terms of the variable $x_{13} = x_1 - x_3$. This gives

$$|\psi\rangle \propto \sum_{x_{23}} \sum_{x_{13}} (1+\exp(-2i\beta x_{23}))(1+\exp[2i\alpha(x_{13}-x_{23})])|x_{23}\rangle|x_{13}\rangle. \tag{19}$$

Proceeding as before, we can find the most probable separations between 1 and 3 by first tracing over x_{23} to give the reduced density matrix, $\rho_{13} = \mathrm{Tr}_{23}\{\rho\}$. The probability distribution is then

$$P(x_{13}) = \langle x_{13}|\rho_{13}|x_{13}\rangle \propto \sum_{x_{23}} \cos^2(\beta x_{23})\cos^2[\alpha(x_{13} - x_{23})]. \tag{20}$$

If we consider the two cosine factors separately, we can find the values of x_{13} that maximize this probability distribution by observation. For a given value of x_{23}, the second cosine factor is maximized for values

$$x_{13} = x_{23} + \frac{n\pi}{\alpha}, \tag{21}$$

where n is an integer. If we were to set the first cosine factor to unity, we see that the sum over all values of x_{23} gives a flat probability distribution, i.e. no value of x_{23} is favoured. However, the first cosine factor is not flat. We can consider it as a weighting factor that enhances the probability for values of $x_{23} = m\pi/\beta$. Combining this observation with (21) means that the most probable separation between particles 1 and 3, $x_{13} = X_{13}$, is

$$X_{13} = \frac{\pi n}{\alpha} + \frac{\pi m}{\beta} = X_{12} + X_{23}, \tag{22}$$

where the last equality follows from our earlier results (10) and (16). This demonstrates the correctness of the transitivity relationship (18) for single scattering events.

4. Multiple scattering events

Now that we have established the mechanism for transitivity, we can consider the case of multiple scattering events. This can be achieved in a straightforward manner by iterating the scheme outlined above. Rau *et al.* have considered multiple scatterings in detail for the case of two particles [5]. They showed that subsequent detections result in a single sharply defined peak in the relative position probability density, i.e. the periodicity of relative localization is removed. In our simulations, we account for both the scattered and non-scattered photons as discussed in section 2.

We can generalize (14) for multiple scatterings from particles 1 and 2, followed by multiple scatterings from particles 2 and 3 as follows,

$$|\psi\rangle \propto \sum_{x_{12}}\left[\prod_j (1 + \exp(2\mathrm{i}\alpha_j x_{12}))\right]|x_{12}\rangle \otimes \sum_{x_{23}}\left[\prod_j (1 + \exp(-2\mathrm{i}\beta_j x_{23}))\right]|x_{23}\rangle, \tag{23}$$

where $\{\alpha_j\} = \{(k\sin\Theta_{12,j})/2\}$ is the sequence of values of detections from 1 and 2, and $\{\beta_j\} = \{(k\sin\Theta_{23,j})/2\}$ is the sequence of values of detections from 2 and 3. If the two terms in square brackets are peaked at $x_{12} = X_{12}$ and $x_{23} = X_{23}$ respectively, it is straightforward to show that the transitivity relation (18) holds for multiple scattering events. This can be seen by calculating the probability distribution, $P(x_{13})$, as before

$$P(x_{13}) \propto \sum_{x_{23}}\prod_j \cos^2(\beta_j x_{23})\cos^2[\alpha_j(x_{13} - x_{23})]. \tag{24}$$

Following the same argument as for the single scattering particle case: for a given value of x_{23}, the second cosine factor has maxima at $x_{13} = X_{12} + x_{23}$, and the first cosine term can be thought of as a weighting factor that enhances the probabilty for $x_{23} = X_{23}$. Combining these results, we see that the most-likely value for x_{13} would be $X_{12} + X_{23}$, as predicted by transitivity. In order to confirm this result for multiple scattering events, we need to carry out numerical simulations of the detection process.

We begin our simulations by scattering 100 photons off particles 1 and 2 and detecting the angle at which they are scattered. This process induces a relative localization between the particles and the pattern of detected photons provides a read-out of the value of the separation that is induced. It is convenient to restrict the separation to some finite interval. For definiteness, we will take $e(x_{12})$ to be $1/(10\lambda)^{1/2}$ for $x_{12} \in [0, 10\lambda]$ and zero otherwise, where λ is the wavelength of the incident photons.

In figure 2(*a*) we have plotted the probability density for the separation of particles 1 and 2 after this process. We see that a single sharp peak has formed, indicating a well-defined relative localization at $X_{12} = 3.1$ in units of the photon wavelength, λ. For different runs of the simulation, this peak is randomly located within the interval shown.

The second step in the simulation involves scattering 100 photons off particles 2 and 3 and recording their scattering angle. We take the initial relative position to be the same as it was initially for particles 1 and 2, i.e. we take $f(x_{23})$ to be $1/(10\lambda)^{1/2}$ for $x_{23} \in [0, 10\lambda]$ and zero otherwise. The probability density for the separation between these two particles after this process is shown in figure 2(*b*). As with particles 1 and 2, a sharp peak emerges which, for different runs of the simulation, is located randomly in the interval shown. The pattern of scattered photons provides a read-out of this location, which in this case is $X_{23} = 5.1\lambda$.

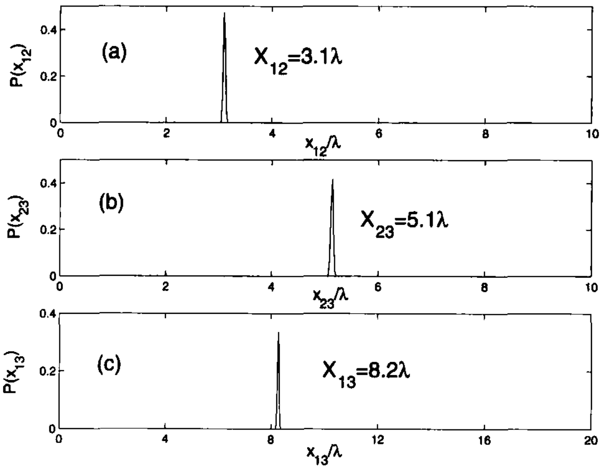

Figure 2. (*a*) The probability density for the relative position between particles 1 and 2 after 100 photons are scattered off them. (*b*) The probability density for the relative position between particles 2 and 3 after 100 photons are subsequently scattered off them. (*c*) The probability density for the relative position of 1 and 3 before any direct measurement is made of this separation.

Finally, we would like to see whether we can predict the outcome of a measurement of the separation between particles 1 and 3 using the results of our measurements of the separations between 1 and 2 and between 2 and 3. Such a prediction relies on the fact that the relative position of 1 and 2 is not disturbed by the measurements on 2 and 3. If the transitivity relationship (18) holds, we would predict a value of $X_{13} = 3.1\lambda + 5.1\lambda = 8.2\lambda$.

In figure 2 (*c*) we have plotted the probability density for the separation between 1 and 3 calculated from the quantum state of the system before any direct measurement of this is made. We see that the distribution is indeed peaked at $X_{13} = 8.2\lambda$, which confirms the prediction of (18). The simulation was repeated many times and, in each case, we were able to predict the value of X_{13} from the other results. This demonstrates the transitivity of the relative positions which are generated by light scattering.

Of course, we need not stop here. We could measure the position of a fourth particle relative to any one of the others and, from the outcome of this, predict the separation between any pair of particles in the system. This process could be continued for many particles. We have seen that these well-defined separations can arise even though each individual particle remains delocalized in absolute space, which suggests that, like the quantum phase of Bose–Einstein condensates [6, 7], the most natural space to work in is relative coordinate space. This shows how a 'classical' framework of relative positions can emerge for a multiparticle quantum system due to entanglement.

5. Multiparticle scattering events

So far, we have considered only scattering from pairs of particles. It is relatively straightforward to generalize this result to include scattering from many particles. One such situation we can consider is that a relative localization, X_{12}, has already been established between particles 1 and 2 (by two particle scattering) and then a third particle, which is delocalized, is added to the system. By scattering light from all three particles, the third particle is localized relative to the other two. We can see this as follows.

We consider that a single photon is scattered from this system and detected at an angle θ† and we do not know which of the particles it scattered from. Following the same formalism as above, the state after the detection can be written in relative coordinates as

$$|\psi\rangle = \frac{1}{3^{1/2}} \sum_{x_{23}} (1 + \exp(\mathrm{i}\zeta X_{12}) + \exp(-\mathrm{i}\zeta x_{23}))|X_{12}\rangle|x_{23}\rangle, \tag{25}$$

where $\zeta = (k \sin\theta)/2$. The probability distribution for the position of particle 2 relative to particle 3, x_{23}, is then

$$\begin{aligned} P(x_{23}) &= \langle x_{23}|\rho_{23}|x_{23}\rangle \\ &= 1 + \frac{2}{3}[\cos(\zeta X_{12}) + \cos(\zeta x_{23}) + \cos(\zeta(X_{12} + x_{23}))], \end{aligned} \tag{26}$$

†In this case, certain detection angles will be preferred even for the first detection since particles 1 and 2 already have a well-defined separation.

where $\rho_{23} = |\psi\rangle\langle\psi|$ in this case. This distribution is no longer flat, but has local maxima at values

$$x_{23} = -\frac{1}{\zeta}\tan^{-1}\left(\frac{\sin(\zeta X_{12})}{1+\cos(\zeta X_{12})}\right) + \frac{2\pi n}{\zeta}, \tag{27}$$

where n is an integer. This means that the detection process begins to localize the third particle relative to particle 2 and hence also to particle 1. As before, we might expect subsequent detections to reinforce this localization. Simulations confirm that this is indeed the case and scattering from all three particles localizes particle 3 relative to the other two.

We can now consider adding a fourth delocalized particle to the system. In this case scattering events between this fourth particle and any one, two, or all three of the other particles will result in relative localizations between every pair of particles. We could also imagine two 'domains' of particles arising with well-defined localizations within themselves but with no relative localization between the domains. Scattering between any number of particles in these two domains would result in relative localizations emerging between all the particles in the two systems. Furthermore, the transitivity result we have demonstrated above (22) ensures that all these relationships are consistent. This result can also be applied to systems for which all the particles are initially delocalized.

6. Conclusion

We have discussed how light scattering from delocalized quantum particles can lead to the emergence of 'classical' relative positions. This process occurs even though the absolute positions of the particles remain undefined and suggests that the natural spatial framework for such a system is relative position. We have then extended this result to systems with more than two particles and showed that the relative localizations induced by either pair-wise or multiparticle scattering events are transitive. This means that relative localizations induced by light scattering are robust and provide a consistent 'classical' coordinate framework.

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