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SUPERDENSE CODING WITH SINGLE-PARTICLE ENTANGLEMENT

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Abstract

Recent work has explored the idea that nonlocality or entanglement involving a single particle should be taken seriously and has real measurable consequences. Theoretical and experimental schemes have shown, for example, that single-particle states can violate Bell's inequalities. Here we discuss the possibility of using single-particle entanglement for implementing a superdense coding protocol. Particle-number superselection rules restrict this scheme to being able to transmit $\log_2(3)$ bits of information. While this falls short of the two-particle limit of two bits, it still exceeds what can be achieved without entanglement.

Keywords: nonlocality, mode entanglement, superdense coding.

1. Introduction

The idea that a single particle can exhibit nonlocality with observable consequences was first put forward in 1991 [1]. Since then, there has been a lot of discussion about the nature of this entanglement, whether it can truly be observed in realistic experiments, and whether it differs in any fundamental way from entangled states consisting of more than one particle [2–8].

A lot of this debate stems from subtleties related to how measurements would be made in an experiment. The standard method for confirming that there is nonlocality in a system is to seek violations of Bell's inequality. This involves sending each particle from an entangled pair to two spatially separated parties (Alice and Bob). These two parties then measure some property of their particle such as spin or polarization, and the two parties later compare their results. Nonlocality (or entanglement) allows correlations that are stronger than anything possible classically. A problem arises, however, if we try to apply the same method to single-particle states. If there is only one particle and (say) Alice measures some property of it, then Bob will have no particle to measure. This means that the two parties cannot correlate their measurement outcomes and so cannot detect nonlocality by this method. Theoretical schemes [9, 10] and experiments [11] have found ways of circumventing this problem by using reference states to supply additional particles to the system. Of course, this opens up these schemes to the criticism that what they are measuring is not truly a single-particle effect. However, by carefully ensuring that the additional particles are added as a part of the local measurement process, it is possible to conclude that they cannot be responsible for any of the nonlocality that is observed.

If single-particle states really can be entangled, then they should be able to be used as a resource for performing quantum information tasks. Specifically, they should enable us to perform tasks beyond

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the capability of states that are not entangled. In this paper, we consider the specific case of superdense coding. We begin by giving a brief overview of superdense coding for two-particle entangled states and then highlight the problems associated with extending it to the case of a single particle. We then propose a protocol that overcomes the problems; however, the trade-off is that we can only transmit one 'trit' (i.e., $\log_2(3)$ bits) of information as opposed to the standard two-particle approach that can transmit two bits. Despite this, our scheme surpasses anything that is possible classically and lends further weight to the notion that single-particle entanglement is not just a mathematical curiosity, but a real effect with measurable consequences.

2. Superdense Coding

Superdense coding is a technique that enables two bits of information to be transmitted between parties by sending only a single qubit between them [12]. This cannot be achieved by classical means and relies on the two parties sharing an entangled state that could, for example, be distributed to them *a priori* by some third party. To take a specific example, suppose Alice and Bob share an entangled state of two spin-1/2 particles of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B\right), \tag{1}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively, denote the spin-up and spin-down states, and Alice has qubit A and Bob has qubit B. Alice then makes one of the four *local unitary* operations $\{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}$ on her qubit, where σ_0 is the identity, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. Finally Alice sends her single qubit to Bob, who is now in possession of the total state. The final states that Bob receives after each of Alice's operations are shown in Table. 1.

Table 1. Final States that Bob Received after Alice's Operation.

Alice's operation	Final state
σ_0	$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$
σ_x	$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$
σ_y	$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle)$
σ_z	$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$

These final states are the four Bell's states and, since they are orthogonal, can all be distinguished by Bob at least in principle. If each of these states represents one piece of information, then Alice has managed to send one out of four messages (i.e., two bits of information) by transmitting only a single qubit. Until recently, experiments have only been able to distinguish three out of these four states [13]. However, more complicated schemes that make use of so-called hyper-entanglement whereby more than one degree of freedom is entangled simultaneously can now distinguish all four states [14].

3. Single-Particle Entanglement

It is interesting to see how superdense coding could be implemented using single-particle entangled states. Suppose that we have a state of the form

$$|\Psi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B\right). \qquad (2)$$

This state is analogous to (1); however, instead of each ket representing the state of a particle, each ket now represents the population of a mode. In this case, we have a single particle in a coherent superposition of being in mode A and mode B. Such a state could be created, for example, by passing a single photon through a 50:50 beam splitter. Each ket then represents the number of photons in each output port of the beam splitter. Although there is only a single particle, this state is still entangled because it is the modes (rather than particles) that are entangled.

Suppose that Alice and Bob share this entangled state and that they want to implement superdense coding. To do this, Alice needs to be able to perform local operations that transform the total state into each of the four Bell's states. To access the $|\Psi^{-}\rangle$ state, Alice needs only to apply a π phase shift to each particle in her mode. This could be achieved, for example, with a phase plate for photons or by adjusting the trapping potential for trapped atoms or ions. The state then becomes

$$|\Psi^{+}\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(-|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B\right) = |\Psi^{-}\rangle. \qquad (3)$$

A problem arises when Alice tries to access the other Bell's states

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B\right), \qquad (4)$$

because these require her to be able to make non-number-conserving operations, i.e., she needs to violate the number-conservation superselection rule. This problem can be circumvented by the use of reference states that act as a reservoir of additional particles. Such an idea was used, for example, in a theoretical proposal for creating and observing a superposition of an atom and a molecule [15]. Such a technique, however, requires that the same reference state is used to create and read out the superposition. Therefore, it cannot be employed in a superdense coding scheme because Alice would need to send the reference state along with her qubit to Bob to enable him to read out the signal. This defeats the purpose of sending two bits of information by sending only a single qubit. We now discuss how this problem can be overcome (at least partially) enabling us to surpass the classical limit for information transfer.

We have already seen that Alice is able to transform from $|\Psi^{+}\rangle$ to $|\Psi^{-}\rangle$ with a simple phase shift. In order to access the other two Bell's states, $|\Phi^{\pm}\rangle$, from $|\Psi^{\pm}\rangle$, Alice could bring in an auxiliary single-particle Fock state, $|1\rangle$, i.e., the total state is then

$$|\Psi^{\pm}\rangle \longrightarrow \frac{1}{\sqrt{2}}|1\rangle_A\left(|1\rangle_A|0\rangle_B \pm |0\rangle_A|1\rangle_B\right). \qquad (5)$$

She could then perform the following transforms:

$$|1\rangle|1\rangle \longrightarrow |2\rangle|0\rangle, \qquad (6)$$

$$|1\rangle|0\rangle \longrightarrow |0\rangle|1\rangle, \qquad (7)$$

which can be implemented with the use of a nonlinear interferometer, as we will discuss in the next section. The state is then

$$\frac{1}{\sqrt{2}}\left(|2\rangle_A|0\rangle_A|0\rangle_B \pm |0\rangle_A|1\rangle_A|1\rangle_B\right). \tag{8}$$

Finally Alice transmits one of her qubits to Bob,

$$\frac{1}{\sqrt{2}}\left(|2\rangle_A|0\rangle_B|0\rangle_B \pm |0\rangle_A|1\rangle_B|1\rangle_B\right). \tag{9}$$

Bob's part of the state now looks just like the Bell's states $|\Phi^\pm\rangle$ that we want. The problem is that it is entangled with the qubit that remains with Alice. She is not able to give this qubit to Bob without violating the spirit of the superdense coding protocol, so if Bob is to make only local measurements, he does not have access to the state of Alice's qubit. This means that he must trace out Alice's qubit and he is left with the mixture

$$\rho = \frac{1}{2}\left(|0\rangle|0\rangle\langle 0|\langle 0| + |1\rangle|1\rangle\langle 1|\langle 1|\right), \tag{10}$$

where we have dropped the subscripts for notational simplicity. This is an equal mixture of Bob having no particles or two particles. Clearly this can readily be distinguished from the one-particle states, $|\Psi^\pm\rangle$, by simple number counting. Bob is also able to distinguish $|\Psi^+\rangle$ and $|\Psi^-\rangle$ by applying a $\pi/2$ phase shift (PS) to one mode and then passing both modes through a 50:50 beam splitter (BS),

$$|\Psi^+\rangle \xrightarrow{\text{PS}} \frac{1}{\sqrt{2}}\left(i|1\rangle|0\rangle + |0\rangle|1\rangle\right) \xrightarrow{\text{BS}} |0\rangle|1\rangle, \tag{11}$$

$$|\Psi^-\rangle \xrightarrow{\text{PS}} \frac{1}{\sqrt{2}}\left(-i|1\rangle|0\rangle + |0\rangle|1\rangle\right) \xrightarrow{\text{BS}} |1\rangle|0\rangle, \tag{12}$$

where we have ignored any irrelevant overall phase. The 50:50 beam splitter transforms the annihilation operators a_1 and a_2 of the two qubits in the following way:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{13}$$

We see from (11) and (12) that, by recording which port the particle emerges from, $|\Psi^+\rangle$ and $|\Psi^-\rangle$ can be distinguished. Overall Bob is able to distinguish three possibilities $|\Psi^+\rangle$, $|\Psi^-\rangle$, and ρ given by (10). This means that Alice can transmit one 'trit' of information (i.e., $\log_2(3) \approx 1.585$ bits) by sending only a single qubit. This falls short of the ultimate two-particle limit of two bits, but it coincides with what is usually achieved in experiments due to the fact that it is impossible to deterministically discriminate all four states using linear optics [13, 16, 17]. An important difference is that recent experiments [14] have shown that it is possible to reach the two-bit limit for two-particle entanglement by making use of hyper-entanglement — whereby quantum systems are simultaneously entangled in more than one degree of freedom. It is not clear how this limit could be achieved with single-particle entanglement, and whether this difference is fundamental requires further investigation. However, it is interesting that the single-particle case still enables us to exceed what is possible without entanglement and is a further demonstration of the existence of single-particle entanglement.

4. Nonlinear Interferometer

This superdense coding scheme relies on Alice being able to perform the transforms given by (6) and (7). These are nontrivial but can be achieved, for example, by the nonlinear interferometer shown in Fig. 1.

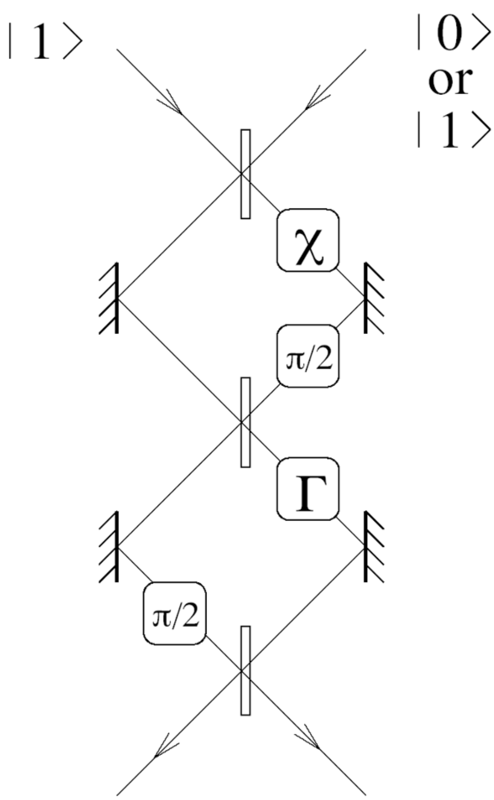

Fig. 1. Nonlinear double interferometer that can be used by Alice to implement the state transforms (6) and (7). Alice combines her qubit from the entangled state with the auxiliary state $|1\rangle$ as the inputs to the interferometer. The three beam splitters are each balanced, i.e., 50:50, χ and Γ represent nonlinearities, and the $\pi/2$ blocks represent $\pi/2$ phase shifts.

In this scheme, Alice combines her qubit from the entangled pair (that she shares with Bob) with the ancillary Fock state, $|1\rangle$, as the inputs to the interferometer. Each of the beam splitters in the scheme is taken to be 50:50 and the boxes containing $\pi/2$ represent $\pi/2$ phase shifts per particle on that path. The boxes containing χ and Γ represent nonlinearities that could be provided, for example, by a nonlinear crystal in the case of photons, or simply by the collisional interactions between particles for atoms. The Hamiltonians associated with these nonlinearities are $H = \chi a^{\dagger 2}a^2$ and $H = \Gamma a^{\dagger 2}a^2$, where a is the annihilation operator for a particle on the path of the interferometer where the nonlinearity acts. The unitary transforms that they perform are $\exp[i\chi n_a(n_a-1)]$ and $\exp[i\Gamma n_a(n_a-1)]$, where $n_a = a^\dagger a$ is the number operator. This means that they simply bring about a phase shift that depends nonlinearly on the number of particles on that path. For our scheme, we take $\Gamma = 2\chi = \pi/2$.

We can directly check that this scheme gives the correct transforms by propagating the initial states through the interferometer. We will use the convention that the first ket represents the state on the left-hand path at each stage of the scheme and the second ket represents the state on the right-hand path. Let us begin with $|1\rangle|0\rangle$,

$$\begin{aligned}
|1\rangle|0\rangle \quad &\xrightarrow{\text{BS}} \quad \frac{1}{\sqrt{2}}\left(i|1\rangle|0\rangle + |0\rangle|1\rangle\right) \xrightarrow{\chi,\pi/2} \frac{i}{\sqrt{2}}\left(|1\rangle|0\rangle + |0\rangle|1\rangle\right) \\
&\xrightarrow{\text{BS}} \quad \frac{e^{i\pi/4}}{\sqrt{2}}\left(|1\rangle|0\rangle + |0\rangle|1\rangle\right) \xrightarrow{\Gamma,\pi/2} \frac{e^{i\pi/4}}{\sqrt{2}}\left(i|1\rangle|0\rangle + |0\rangle|1\rangle\right) \\
&\xrightarrow{\text{BS}} \quad e^{i3\pi/4}|0\rangle|1\rangle, \qquad (14)
\end{aligned}$$

which is the result we want. Of course, there are much simpler ways of performing this transform if that is all we need. For example, the nonlinearities play no role in this case because there is only ever at most one particle on each path. However, we need our interferometer to also simultaneously carry out the other transform (7) and, in this case, the nonlinearities play an important role,

$$\begin{aligned} |1\rangle|1\rangle \quad &\xrightarrow{\text{BS}} \quad \frac{i}{\sqrt{2}}\left(|2\rangle|0\rangle + |0\rangle|2\rangle\right) \xrightarrow{\chi,\pi/2} \frac{i}{\sqrt{2}}\left(|2\rangle|0\rangle - i|0\rangle|2\rangle\right) \\ &\xrightarrow{\text{BS}} \quad \frac{e^{i3\pi/4}}{2}\left(-|2\rangle|0\rangle + |0\rangle|2\rangle + \sqrt{2}|1\rangle|1\rangle\right) \\ &\xrightarrow{\Gamma,\pi/2} \quad \frac{e^{i3\pi/4}}{2}\left(|2\rangle|0\rangle - |0\rangle|2\rangle + i\sqrt{2}|1\rangle|1\rangle\right) \\ &\xrightarrow{\text{BS}} \quad -e^{i3\pi/4}|2\rangle|0\rangle, \end{aligned} \tag{15}$$

which again gives the transform we want — the overall phases do not affect our scheme. This confirms that the interferometer shown in Fig. 1 successfully makes the transforms we require.

5. Full Scheme

Everything can now be combined into a full superdense coding scheme as shown in Fig. 2. A third party (Charlie) begins by creating the single-particle entangled state,

$$\frac{1}{\sqrt{2}}\left(i|1\rangle|0\rangle + |0\rangle|1\rangle\right), \tag{16}$$

by passing a single photon through a 50:50 beam splitter. He then sends one mode to Alice and the other to Bob. Alice then performs one of three operations on her qubit corresponding to one of the three possible pieces of information she wants to send. These operations, which are represented by a black-box in Fig. 2, are: do nothing (identity), a π-phase shift, and the transform outlined in Sec. 4. above. Alice then passes a single qubit to Bob, shown as the lower path in Fig. 2. Finally Bob combines the qubit that Alice gives him with his own qubit at a 50:50 beam splitter and detects the number of particles in each output port. There are three possible measurement outcomes:

1. Bob detects a single particle at the upper detector;
2. Bob detects a single particle at the lower detector;
3. Bob detects either zero or two particles at the output detectors.

Each of these correspond to one of the three messages that Alice was able to send.

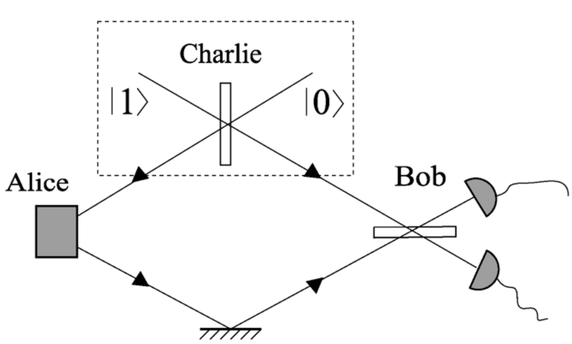

Fig. 2. Full superdense coding scheme for single-particle states. Charlie creates a single-particle entangled state by passing a single particle through a 50:50 beam splitter and then distributes this state to Alice and Bob. Alice makes one of three local unitary transforms on her qubit (as discussed in the text) represented in the diagram by her 'black box'. She then passes her qubit to Bob, who combines it with his qubit at a 50:50 beam splitter. By detecting the output from this beam splitter, Bob can unambiguously determine which of the three operations Alice performed.

6. Discussion

A natural question that arises is whether it is possible to improve this scheme so that it is able to distinguish four possible messages, i.e., two bits of information. This certainly would be possible if Alice were allowed to pass her ancillary state to Bob as well as the other qubit. However, this defeats the purpose of superdense coding since now two qubits would be passed between parties and it should not surprise us at all that two bits of information are able to be transferred. Indeed this could be achieved without any entanglement.

Instead, Alice must retain the ancillary qubit that is created when the transform in Sec. 4. is performed. The question then becomes — can Alice perform local operations that wash out which way the information is contained in this ancillary qubit, i.e., can she disentangle her qubit from the rest of the state? We can easily see from the following example that this cannot be possible since it would allow superluminal communication. Suppose Alice and Bob share the state (1),

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B\right). \tag{17}$$

Alice can convert this to $|\Psi^-\rangle$ with a local π-phase shift on her qubit. Now let us assume that he is able to disentangle her qubit from the rest of the state by local operations. This would give a final state

$$|\eta\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|\downarrow\rangle_B \pm |\uparrow\rangle_B\right), \tag{18}$$

where $|\eta\rangle_A$ is the (disentangled) qubit that Alice is left with. Now Bob should be able to distinguish the two possibilities for his state $\left(|\downarrow\rangle_B \pm |\uparrow\rangle_B\right)/\sqrt{2}$, which suggests that Alice could communicate with Bob without ever sending a qubit to him, i.e., superluminal communication. We know that this is not possible and so clearly our assumption that Alice can disentangle her qubit from the rest of the system in this way is not valid.

7. Conclusions

We have proposed a simple scheme for implementing a superdense coding protocol using single-particle entanglement. Subtleties relating to number-conservation superselection rules make the scheme more complicated than two-particle entangled state schemes. They also mean that this scheme is limited to transferring $\log_2(3)$ bits of information per qubit sent between the parties. Whether different schemes that employ other ideas such as hyper-entanglement allow the ultimate two-bit limit to be achieved remains to be seen. That aside, the present scheme is able to surpass the limit of what can be achieved in the absence of entanglement and gives further support for the idea that single-particle entanglement is real and has observable consequences.

Acknowledgments

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