Entanglement and nonlocality of a single relativistic particle

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Recent work has argued that the concepts of entanglement and nonlocality must be taken seriously even in systems consisting of only a single particle. These treatments, however, are nonrelativistic, and, if single-particle entanglement is fundamental, it should also persist in a relativistic description. Here, we consider a spin-1/2 particle in a superposition of two different velocities as viewed by an observer in a relativistically boosted inertial frame and show that the entanglement between the two velocity modes survives right up to the speed of light. We also discuss how quantum gates could be implemented in this way and apply our results to the case of a superconductor. In particular, we show that an *s*-wave superconductor would have *p*-wave components for a boosted observer.

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Entanglement and its related nonlocality are believed to be the root cause of all the major differences between quantum and classical physics. At present, however, nature requires two different theories to be combined in order to reach a satisfactory model of reality. Relativity is as important and well-tested in its own domain as quantum mechanics and only the marriage of the two—known as quantum field theory—yields experimentally satisfactory results. It is therefore paramount that entanglement is analyzed from the relativistic perspective. Here, we show that the nonrelativistic concept of single-particle entanglement [1,2] survives in quantum field theory and that boosted observers will see a single-particle violation of Bell's inequalities. We prove, however, that the amount of entanglement is dependent on the inertial frame. Though the relativistic correction to entanglement is small at small speeds, this effect may play an important role in the future of quantum information processing. We also consider the specific case of a superconductor as viewed by a boosted observer and show that, to the observer, an *s*-wave superconductor acquires *p*-wave components.

Imagine that we have a massive particle of spin s moving with velocity v_1, which is viewed by a relativistic observer traveling at velocity v_2. If the two velocities are not collinear, the overall transform is not simply a Lorentz boost, but also involves a rotation that depends on the values of v_1 and v_2. More specifically, if we perform a boost in the x direction followed by a boost in the y direction, a rotation will result about the z axis. The unitary matrix representing this rotation was worked out by Wigner in a seminal paper in 1939 [3].

Unitary representations of Lorentz boosts and Wigner's rotations are part of the common folklore of quantum field theory and we need not explain them in detail here; an excellent treatment can be found in [4]. We will instead begin by considering the effect of Wigner rotations on single-particle states. For this, we need to know that a general state of a single particle,

$$|\Psi\rangle = \int d\mu(v) f(v) |v\rangle |\chi\rangle, \tag{1}$$

where $d\mu$ is a relativistically invariant integration measure, $|v\rangle$ is the ket representing velocity, $|\chi\rangle$ is the ket representing spin, and $f(v)$ is the velocity space wave function, will transform under a general Lorentz boost in the following way:

$$U(\Lambda)|\Psi\rangle = \int d\mu(v) f(\Lambda^{-1}v) |v\rangle D(W(\Lambda)) |\chi\rangle, \tag{2}$$

where Λ is the Lorentz boost and $D(W(\Lambda))$ is the unitary transformation representing the Wigner rotation W that itself is a function of the boost. We will discuss the form of D below.

In previous work [5], we argued that the notion of single-particle nonlocality (and entanglement) should be taken seriously. To put it simply, a single particle existing in a superposition of two spatially distinct locations can violate Bell's inequalities in much the same way that the usual two particle Einstein-Podolsky-Rosen (EPR) (or Bohm) state does. The main subtlety in this argument was that certain operations needed to be performed, which appeared to contradict superselection rules. We do not wish to restate our arguments here, but, in short, a careful choice of ancillary systems allows us to sidestep any superselection restrictions. The reader interested in a more detailed argument is encouraged to consult our discussion in [5,6].

We would now like to directly investigate the effects of Lorentz boosts on single-particle entanglement and nonlocality. To put it more physically, if one observer records in his experiments a violation of a Bell inequality due to single-particle entanglement, will this also be true for all other inertial observers? Or, can one observer see something as entangled that appears disentangled to another observer who moves uniformly with respect to him?

Standard two particle entanglement has been studied a number of times with respect to relativity in inertial [7–10] and accelerated [11] frames as well as the transform of a single spin-1/2 particle [12]. However, to our knowledge, no one has ever considered the case of a single-particle (mode) entanglement. The importance of the latter is twofold. First, the relativistic behavior is most transparent for single-particles and any many-particle treatment follows straightforwardly by direct iteration of the single-particle formalism.

Second, it would be hard to argue that mode entanglement was genuine if it were only present in standard nonrelativistic quantum mechanics.

For definiteness, we begin by considering the simple case of a single-particle state, where the particle is moving in two opposing directions along the y axis with equal amplitudes and speeds, v_1. This can be written in terms of the populations of the two velocity modes v_1 and $-v_1$ as

$$\frac{1}{\sqrt{2}}(|1\rangle_{v_1}|0\rangle_{-v_1}+|0\rangle_{v_1}|1\rangle_{-v_1}). \tag{3}$$

This state is analogous to the single-particle dual-rail encoding of two distinct *spatial* modes used, for example, in the Knill-Laflamme-Milburn (KLM) scheme for linear optical quantum computing [13] and can be thought of as having entanglement between the velocity modes [5,6]. It is convenient for us to represent it in the equivalent form of a coherent superposition, $(|v_1\rangle+|-v_1\rangle)/\sqrt{2}$, although this form somewhat obscures the entanglement present. We will consider the case of a spin-1/2 particle with two possible z components of spin: spin-up, $|\uparrow\rangle$, and spin-down, $|\downarrow\rangle$. The spin of the particle is taken to initially point up and is not entangled with the velocity. This state can be written as

$$|\Psi\rangle=\frac{1}{\sqrt{2}}\int dv_1 f(v_1)(|v_1\rangle+|-v_1\rangle)|\uparrow\rangle, \tag{4}$$

where the shape of the wavepacket in velocity space is given by $f(v_1)$. States of this form could, for example, be created by exciting an electron into the conduction band in graphene. In this case, v_1 is a small but appreciable fraction of the speed of light, which is important for being able to detect relativistic effects [14]. Superpositions of different velocities can be achieved by scattering from impurities. We can think state (4) as initially having entanglement between the velocity modes v_1 and $-v_1$, but no entanglement between velocity and spin. We will discuss both these types of entanglement and the relationship between them.

We now consider an observer boosted in a direction orthogonal to the velocity of the particle, i.e., in the x-z plane. For a boost of velocity v_2 at an angle ϕ in the x-z plane (see Fig. 1), the transform D is given by

$$D=\sigma_0\cos\omega+i\sin\omega(\cos\phi\sigma_x-\sin\phi\sigma_z), \tag{5}$$

where σ_0 is the identity operator and σ_x and σ_z are the Pauli spin operators. The angle of Wigner's rotation is given by,

$$\sin\omega=\sqrt{\frac{(\gamma_1-1)(\gamma_2-1)}{2(1+\gamma_1\gamma_2)}}, \tag{6}$$

with $\gamma_{1,2}=[1-(v_{1,2}/c)^2]^{-1/2}$ and the axis of rotation is given by $\hat{n}=\widehat{v_2}\times\widehat{v_1}$. This means that ω has different signs for v_1 and $-v_1$. The boosted observer will therefore see state (4) as

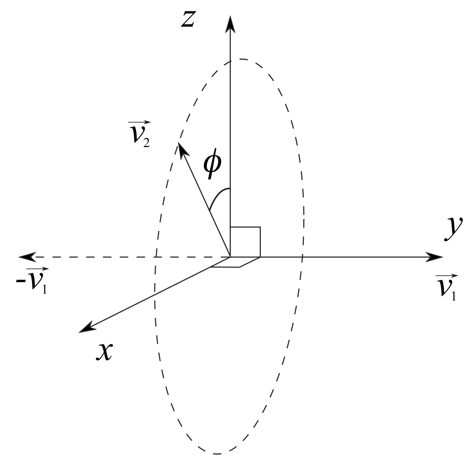

FIG. 1. A single spin-1/2 particle initially has its spin pointing up in the z direction. It is boosted to some velocity v_1 in the y direction (or a superposition of velocities $+v_1$ and $-v_1$ along the y axis). The observer of the system is moving at velocity v_2 perpendicular to the velocity of the particle, i.e., in the x-z plane. The direction of the observer's boost is given by the angle ϕ from the z axis.

$$\begin{aligned}|\Psi'\rangle=&\frac{1}{\sqrt{2}}|v_{1+}\rangle[(\cos\omega-i\sin\omega\sin\phi)|\uparrow\rangle+i\sin\omega\cos\phi|\downarrow\rangle]\\&+\frac{1}{\sqrt{2}}|v_{1-}\rangle[(\cos\omega+i\sin\omega\sin\phi)|\uparrow\rangle\\&-i\sin\omega\cos\phi|\downarrow\rangle],\end{aligned} \tag{7}$$

where we have taken the wave packet to be a delta function centered at v_1, for the sake of simplicity. In this case, the velocity states are transformed to new values for the boosted observer: $v_1\rightarrow v_{1+}$ and $-v_1\rightarrow v_{1-}$. The overall entanglement in this state (spin and velocity) remains the same however the entanglement between the velocity degrees of freedom only (or between velocity and spin) can change.

Let us first consider just the velocity degrees of freedom. Since we are not interested in spin, we can trace it out and the resulting density matrix is

$$\begin{aligned}\rho'=&\frac{1}{2}(|v_{1+}\rangle\langle v_{1+}|+|v_{1-}\rangle\langle v_{1-}|)\\&+\frac{1}{2}[(\cos(2\omega)-i\sin(2\omega)\sin\phi)|v_{1+}\rangle\langle v_{1-}|\\&+(\cos(2\omega)+i\sin(2\omega)\sin\phi)|v_{1-}\rangle\langle v_{1+}|],\end{aligned} \tag{8}$$

where $\cos(2\omega)$ has the simple form

$$\cos(2\omega)=\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}. \tag{9}$$

From this it is clear that the factor $\cos 2\omega$ determines the degree of the reduction in the off-diagonal elements, and hence the decoherence of entanglement. When $\phi=\pi/2$, the magnitude of the off-diagonal elements is always 1/2, regardless of ω and so the velocity modes remain maximally entangled (or equivalently, the velocity and spin components

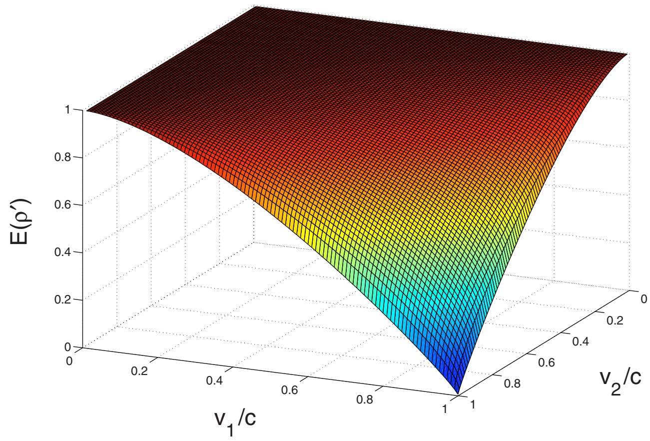

FIG. 2. (Color online) The relative entropy of entanglement for ρ' given by Eq. (8) as a function of the velocities v_1 and v_2.

never become entangled). The other extreme case is $\phi=0$, for which Eq. (8) reduces to,

$$\rho' = \frac{1}{2}(|v_{1+}\rangle\langle v_{1+}| + |v_{1-}\rangle\langle v_{1-}| + \cos(2\omega)|v_{1+}\rangle\langle v_{1-}| + \cos(2\omega)|v_{1-}\rangle\langle v_{1+}|), \tag{10}$$

In the limit of small velocities we have $\cos 2\omega \rightarrow 1$ and so the velocity components are maximally entangled. In the opposite limit, i.e., both velocities approach the speed of light, we have $\gamma_1, \gamma_2 \rightarrow \infty$, which means $\cos 2\omega \rightarrow 0$ and the velocity components are disentangled (or equivalently velocity and spin are maximally entangled).

Another way to understand this is to calculate the relative entropy of entanglement for the state ρ' when $\phi=0$. A simple calculation yields

$$E(\rho') = 1 - S(\rho'), \tag{11}$$

where $S(\rho')=-\text{tr}\{\rho' \log_2 \rho'\}$ is the von Neumann entropy of the state ρ' given by Eq. (8). Here, $S(\rho')$ is a measure of entanglement between the spin and velocity components and $E(\rho')$ is a measure of the entanglement between the velocity modes. Expression (11) shows that as the entanglement between velocity and spin increases, the entanglement between the velocity components decreases, as we would expect. $E(\rho')$ is given by,

$$E(\rho') = 1 + \left(\frac{1+\cos 2\omega}{2}\right)\log_2\left(\frac{1+\cos 2\omega}{2}\right) + \left(\frac{1-\cos 2\omega}{2}\right)\log_2\left(\frac{1-\cos 2\omega}{2}\right). \tag{12}$$

A plot of $E(\rho')$ versus v_1 and v_2 is shown in Fig. 2. For small values of v_1 and v_2, the velocity components of the state are highly entangled and, for large values, they become disentangled. Interestingly, the entanglement only vanishes when v_1 and v_2 are both equal to the speed of light. In other words, for massive particles the velocity components of the state will always appear entangled regardless of the boost applied to the observer.

We can also calculate the degree of violation of the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequalities. This is important since it is a common route to measuring the degree of entanglement in one or two particle systems. Here, we rely on the result of the Horodecki family given in [15]. The value of the Bell operator turns out to be

$$B = 2\sqrt{1+\cos^2 2\omega}. \tag{13}$$

Violations of the CHSH inequality are possible for states for which $B>2$. The form of Eq. (13) means that violations should be observable for our single-particle state (4) for all observers right up to (but not including) the speed of light. Our analysis clearly shows that single-particle entanglement is a genuine feature of quantum field theory and survives the introduction of relativity.

It is worth pausing to discuss how these entanglement effects could, in principle, be measured. We have seen that the factor $\cos(2\omega)$ contains all the information about the entanglement and so it would be a very useful quantity to measure. One way this could be achieved is to scatter two relativistically boosted spin-1/2 particles. Suppose, for example, we have two particles traveling at velocities v_1 and v_2, respectively, in the y and z directions and that both initially have their spins pointing in the z direction (see Fig. 3). These particles are then scattered from one another—shown schematically in Fig. 3 as passing them through a 50:50 beam splitter. In the absence of a Wigner rotation, the Pauli exclusion principle would ensure that one particle is always detected at each output port, since both particles are in the same spin state. In reality, each particle will see the other as being rotated and therefore having some component with the opposite spin. This means that there will be some probability that both particles can be detected at the same output port and that this probability depends on the angle of the Wigner rotation. A straightforward calculation reveals that the probability of detecting both particles at the same port is given by $\sin^2(\omega)/2$. Measuring this quantity reveals $\cos(2\omega)$ and hence also the entanglement properties of the system.

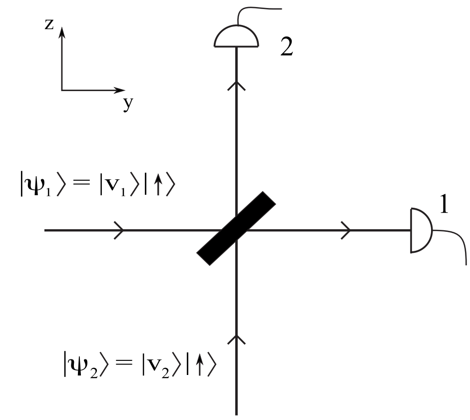

FIG. 3. Scheme for measuring the Wigner rotation. Two relativistically boosted spin-1/2 particles (in the same spin state) are scattered or passed through a beam splitter. The probability that both particles are detected at the same output port is related to the angle ω of the Wigner rotation.

Entanglement transforms for relativistic observers have interesting consequences. Boosts, for example, can give rise to controlled operations between the spin and velocity qubits. To take a specific case, when $\phi=0$ and v_1 and v_2 approach the speed of light, we have $\sin(\omega)=\cos(\omega)=1/\sqrt{2}$, and so the initial state is seen by the observer as,

$$|\Psi'\rangle = \frac{1}{2}|v_{1+}\rangle(|\uparrow\rangle + i|\downarrow\rangle) + \frac{1}{2}|v_{1-}\rangle(|\uparrow\rangle - i|\downarrow\rangle). \quad (14)$$

This is a controlled operation similar, for example, to a controlled-NOT (CNOT) operation since, by boosting to the speed of light, the spin state $|\uparrow\rangle$ is transformed to one of two orthogonal states controlled by the velocity state.

It is interesting to investigate how our results generalize to relativistic quantum systems containing more than one particle. One interesting physical example is how superconducting Cooper pair states are transformed by relativistic boosts. An s-wave Cooper pair state [16] can be written as

$$|\Psi\rangle = (|v_1,-v_1\rangle + |-v_1,v_1\rangle)(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle). \quad (15)$$

The term in the first set of brackets is the velocity (or momentum) component of the Cooper pair and the term in the second set is the spin component. In each case, the first and second parts of each ket correspond to electrons 1 and 2, respectively. This has an overall spin of zero and so is in a singlet (or s-wave) state. To an observer in a frame boosted to a velocity v_2 in the plane orthogonal to v_1 (see Fig. 1), this state is transformed to,

$$\begin{aligned}|\Psi'\rangle = {} & \cos(2\omega)(|v_{1+},v_{1-}\rangle + |v_{1-},v_{1+}\rangle)(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle) \\ & + i\sin(2\omega)(|v_{1+},v_{1-}\rangle - |v_{1-},v_{1+}\rangle) \\ & \times[\cos\phi(|\downarrow,\downarrow\rangle - |\uparrow,\uparrow\rangle) - \sin\phi(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)]. \end{aligned} \quad (16)$$

The transformed state is now a superposition of the initial s-wave singlet with a p-wave triplet [17], the components of which are determined by ϕ. In the extreme limit that both v_1 and v_2 approach the speed of light, the s-wave superconductor is completely transformed into a p-wave state. Recent experiments [18] have confirmed the existence of p-wave symmetry and the pairing of triplet spins in the superconductor strontium ruthenate (Sr_2RuO_4) in an unboosted frame.

This is an interesting result because it suggests a resolution to an apparent problem. We would not expect a system to appear superconducting to some observers but not to others. For example, we would expect a levitating magnet to appear as such to all observers. However, we also know that the nature of entanglement depends on the frame of the observer and that superconductivity is due to the entanglement of Cooper pairs. This result explains how both can be true—although the entangled state is transformed, it changes to another superconducting state to preserve the overall superconductivity for all observers. It would be interesting to consider how other physical effects that depend on entanglement appear in different inertial frames.

Our results not only show that single-particle entanglement persists in relativistically boosted frames, but also suggests that it may be a useful tool for investigating more complicated systems. By iterating the transforms we have discussed for single particles, we may be able to gain insight into how multiparticle systems appear to boosted observers. This may prove to be a valuable tool for developing a general quantum field theory description of entanglement.

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