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# Quantum state of a trapped Bose–Einstein condensate

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## Abstract

The quantum state of a single symmetry-broken condensate at zero temperature is calculated using perturbative techniques. For a fixed mean number of atoms, the state is found to closely approximate a number squeezed state. We propose a means of experimentally testing this state, based on the periodic collapses and revivals of its phase. © 1998 Elsevier Science B.V.

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The experimental creation of Bose–Einstein condensates [1,2] has sparked great interest in their properties. One of the most important and interesting questions is what the quantum state of the condensate is.

A calculation of this state forms the main focus of this Letter. As a secondary issue, we consider how the state evolves with time, due to diffusion. In this second part, we apply known techniques to demonstrate how an investigation of diffusion times could be used to test our prediction of the state.

The first calculation of the state was provided by Lewenstein and You [3] and predicted an amplitude eigenstate. However, only terms of up to quadratic order were retained in their calculation. Here, we provide a more rigorous treatment by retaining terms of all orders. For an assumption of Gaussian statistics, we predict a strongly amplitude squeezed state. When we include the effects of the non-Gaussian terms by

means of a perturbation, we find that this state is “bent” into something very close to a number squeezed state.

The state of the condensate will depend critically on how it is prepared. We consider an idealized system consisting of two condensates where one is very much larger than the other and is treated simply as a phase reference. Furthermore, we take the damping and pumping rates to be vanishingly small compared to the trap frequencies and collision rates. The consequence of this is that the condensate remains in thermodynamic equilibrium throughout its preparation. Our calculations are all for a condensate at zero temperature, and the state we seek is that of a single symmetry-broken condensate<sup>2</sup>.

In our calculation, we break the symmetry by introducing a driving term of which we then take the vanishing limit. We do not, however, propose that such a coherent driving term is real. Here it is simply a math-

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<sup>2</sup> The Zeno effect prevents phase diffusion during the preparation. This is because, over sufficiently short time intervals, the driving and damping, which are linear in time, beat the diffusion, which is quadratic in time.

emathical tool to select a single phase by comparison with the reference condensate. In the thermodynamic limit, the true quantum state will be a mixture, over all possible phases, of the state we find.

We present results here for a condensate of  $^{87}\text{Rb}$  atoms in an isotropic harmonic trap with frequency,  $\omega/2\pi = 60$  Hz. We take the mean number of atoms in the condensate to be  $\bar{N} = 2000$ . These parameters closely match those of an experiment conducted by Anderson et al. [1].

A full calculation would demand a treatment of all the non-condensate modes. However, it can be shown that, at zero temperature, these modes have a negligible effect on the statistics of the condensate [4]. We therefore consider only the condensate mode. The free energy for this system is

$$F = (\hbar\tilde{\omega} - \mu)a^\dagger a + \hbar K a^{\dagger 2} a^2 + i\hbar(a^\dagger E - aE^*) + a^\dagger \Gamma + a\Gamma^\dagger, \quad (1)$$

where  $a$ ,  $a^\dagger$  are the condensate mode operators,  $\mu$  is a chemical potential which constrains the mean number of atoms,  $E$  is a driving field, and  $\Gamma$  is a coupling to a vacuum bath, which is appropriate since the system is at zero temperature. We are able to neglect the higher order modes, even though we are discussing fluctuations, since our description is of quasiparticles. At finite temperatures, we would need to account for real particle fluctuations. In such a case, we would of course need to retain higher order modes. In the present calculation, we retain the leading order effect of the fluctuations and have shown that higher order terms are very small in comparison.

We take the limit in which the driving and damping vanish in such a way that the coherent amplitude remains finite. The parameters  $\tilde{\omega}$  and  $K$  can be found in terms of the condensate wave function,  $\Psi_c$ , as follows<sup>3</sup>,

$$\tilde{\omega} = \int d^3r \Psi_c^*(\mathbf{r}) \left( \frac{\hbar}{2M} \nabla^2 + \frac{1}{2\hbar} M \omega^2 r^2 \right) \Psi_c(\mathbf{r}), \quad (2)$$

$$K = \frac{2\pi\tilde{a}\hbar}{M} \int d^3r \Psi_c^*(\mathbf{r}) \Psi_c^*(\mathbf{r}) \Psi_c(\mathbf{r}) \Psi_c(\mathbf{r}), \quad (3)$$

<sup>3</sup> Strictly,  $K$  is not constant since the condensate mode changes as atoms are added to it. However, since the final state is strongly number squeezed, a good approximation is to take  $K$  corresponding to the fixed mean number of atoms in the condensate.

where  $\tilde{a}$  is the S-wave scattering length and  $M$  is the atomic mass. For  $^{87}\text{Rb}$ ,  $M = 87$  amu and  $\tilde{a} \approx 100 a_0$  [5] where  $a_0$  is the Bohr radius. The form of the interaction term rests on the standard assumption of point collisions. Techniques for determining  $\Psi_c$  are well established [6]. For the parameters considered here, the value of  $K$  is about  $0.38 \text{ s}^{-1}$ .

We can rewrite the condensate mode operators in terms of a coherent amplitude and a fluctuation,  $a \rightarrow \alpha + \delta a$ . We choose the phase so that  $\alpha$  is real. The condensate state will be the lowest energy eigenstate of the free energy. Our approach is to find the eigenstates for a Gaussian approximation and then to treat the non-Gaussian terms as a perturbation. We begin by splitting  $F$  into a Gaussian part,  $F_0$ , and a non-Gaussian part,  $F'$ . The criteria which these two components must fulfill are

$$F = F_0 + F', \quad (4)$$

$$\langle F' \rangle = 0, \quad (5)$$

$$\langle [\delta a, F'] \rangle = \langle [\delta a^\dagger, F'] \rangle = 0, \quad (6)$$

$$\langle [\delta a, [\delta a, F']] \rangle = \langle [\delta a, [\delta a^\dagger, F']] \rangle = 0. \quad (7)$$

A straightforward calculation yields

$$F_0 = \text{const} + Q\delta a^2 + P\delta a^\dagger \delta a + Q^* \delta a^{\dagger 2}, \quad (8)$$

where

$$P = (\hbar\tilde{\omega} - \mu) + 4\hbar K (\alpha^2 + \langle \delta a^\dagger \delta a \rangle), \quad (9)$$

$$Q = \hbar K \alpha^2 + \hbar K \langle \delta a^{\dagger 2} \rangle. \quad (10)$$

$$F' = \hbar K [\delta a^{\dagger 2} \delta a^2 + 2\alpha (\delta a^\dagger \delta a^2 + \delta a^{\dagger 2} \delta a) - (\langle \delta a^2 \rangle \delta a^{\dagger 2} + 4\langle \delta a^\dagger \delta a \rangle \delta a^\dagger \delta a + \langle \delta a^{\dagger 2} \rangle \delta a^2) - \alpha (\langle \delta a^2 \rangle + 4\langle \delta a^\dagger \delta a \rangle + \langle \delta a^{\dagger 2} \rangle) (\delta a + \delta a^\dagger) + \langle \delta a^2 \rangle \langle \delta a^{\dagger 2} \rangle + 2\hbar K \langle \delta a^\dagger \delta a \rangle^2]. \quad (11)$$

We have neglected the damping and driving parts since these are taken to be vanishingly small. The unperturbed free energy,  $F_0$ , can be diagonalized by making a transformation of the form

$$\delta a = (\cosh r) \delta b + e^{-2i\phi} (\sinh r) \delta b^\dagger. \quad (12)$$

It can be shown that the free energy is minimized when  $\phi = 0$ . Therefore, the state we seek has no correlation

between the quadratures  $\delta X = \delta a + \delta a^\dagger$  and  $\delta Y = -i(\delta a - \delta a^\dagger)$ . This allows us to write

$$\langle \delta a^{\dagger 2} \rangle = \langle \delta a^2 \rangle = \frac{1}{4} \left( \langle \delta X^2 \rangle - \frac{1}{\langle \delta X^2 \rangle} \right), \quad (13)$$

$$\langle \delta a^\dagger \delta a \rangle = \frac{1}{4} \left( \langle \delta X^2 \rangle + \frac{1}{\langle \delta X^2 \rangle} - 2 \right), \quad (14)$$

where we have used the result  $\langle \delta X^2 \rangle \langle \delta Y^2 \rangle = 1$  for a pure Gaussian state. This assumption of Gaussian statistics is a perfectly valid one to make since, at this point, we are diagonalizing only the Gaussian part of the free energy. We can solve for this part precisely. Our Gaussian assumption at this stage will, in no way, make our later non-Gaussian perturbations invalid. A relaxation of the pure state assumption is currently under investigation.

The driving strength,  $E$ , and the chemical potential,  $\mu$ , are found by setting the first time derivatives of  $\langle a \rangle$  and  $\langle a^\dagger a \rangle$  equal to zero, using Eq. (1). These conditions are true only during the state preparation. Obviously they do not hold later when we come to investigate how the condensate evolves with time. The solutions are

$$E = \frac{\gamma}{2\alpha} (\alpha^2 + \langle \delta a^\dagger \delta a \rangle), \quad (15)$$

$$\mu = \hbar\tilde{\omega} + 2\hbar K \left( \alpha^2 + \langle \delta a^2 \rangle + 2\langle \delta a^\dagger \delta a \rangle + \frac{1}{\alpha} \langle \delta a^\dagger \delta a^2 \rangle \right) + \frac{i\hbar\gamma}{2\alpha^2} \langle \delta a^\dagger \delta a \rangle. \quad (16)$$

The damping rate,  $\gamma$ , is defined by  $\gamma \equiv \rho^2(\omega)g^2(\omega)$ , where  $\omega$  is the frequency of the condensate mode,  $\rho$  is the bath density of states function, and  $g$  is the coupling function between the condensate mode and the bath [7].

We are interested in the limit  $E, \gamma \rightarrow 0$ . We can find the coupling parameter,  $r$ , by substituting (12) into (8) and setting the coefficients of  $\delta b^2$  and  $\delta b^{\dagger 2}$  to zero. For  $\alpha^2 \gg 1$ , this gives

$$\exp(2r) = \langle \delta X^2 \rangle = \sqrt{\frac{P-2Q}{P+2Q}} \approx \left( \frac{1}{2\alpha} \right)^{2/3}. \quad (17)$$

With this substitution, (8) takes the diagonal form

$$F_0 = \text{const} + \hbar K \sqrt{P^2 - 4Q^2} \delta b^\dagger \delta b. \quad (18)$$

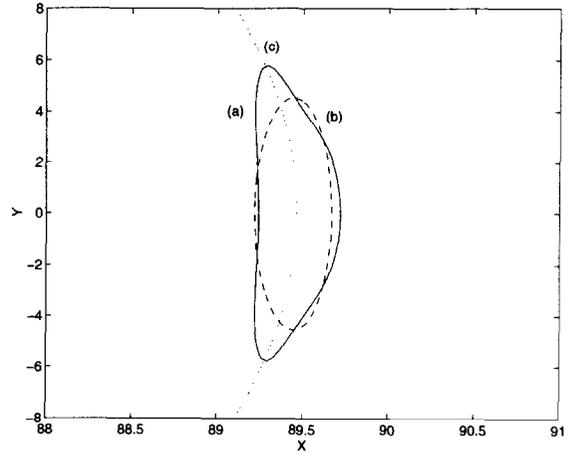


Fig. 1. Error contour of the Wigner function for (a) the condensate state with 2000 atoms, (b) the unperturbed state, and (c) part of the constant intensity line for comparison.

The eigenstates of this free energy are clearly the number states in the  $\delta b$  representation,  $\{|i\rangle\}_{i=0}^\infty$ . If we transform back to the  $\delta a$  representation, it can be seen from the form of the transformation (12) that these states are  $|\tilde{i}\rangle_{\delta a} = S(r)|i\rangle$ , where  $S(r)$  is the familiar squeeze operator,  $S(r) = \exp[r(\delta a^2 - \delta a^{\dagger 2})/2]$ .

Finally, we reintroduce the coherent amplitude,  $\alpha$ , by means of the displacement operator,  $D(\alpha) = \exp(\alpha\delta a^\dagger - \alpha^*\delta a)$ . The eigenstates of the Gaussian part of (1) are therefore

$$|\tilde{i}\rangle \equiv |\tilde{i}\rangle_a = D(\alpha)S(r)|i\rangle. \quad (19)$$

For clarity, the notation  $|\tilde{i}\rangle$  represents the number state  $|i\rangle$  operated on firstly by the squeeze operator  $S(r)$ , where  $r$  is given by (17), then by the displacement operator  $D(\alpha)$  for the appropriate coherent amplitude.

The condensate state is the eigenstate with lowest energy, namely  $|\tilde{0}\rangle$ . This is simply the amplitude squeezed state with  $\langle \delta X^2 \rangle = (2\alpha)^{-2/3}$ . It can be shown that this corresponds to the squeezed state with the minimum possible number fluctuations for a given coherent amplitude. The one standard deviation error contour of the Wigner function of this state is shown in Fig. 1.

We see that retaining terms of all orders in the fluctuation operators has the important consequence of producing a finite width in the  $X$ -quadrature. Calculations which neglect terms of higher than quadratic order give rise to the unphysical result of an amplitude

eigenstate since the chemical potential precisely cancels the  $Y$ -dependence of the free energy to quadratic order. The higher order terms need to be treated so that at least the leading order  $Y$ -dependence is included.

We now wish to relax the assumption of Gaussian statistics. We achieve this by treating the non-Gaussian part of the free energy,  $F'$ , as a perturbation to the state  $|\tilde{0}\rangle$ . Using standard time-independent, non-degenerate perturbation theory, the state of the condensate, including a first-order perturbation, is

$$|\Phi^{(1)}\rangle = |\tilde{0}\rangle + \sum_{k \neq 0} \frac{F'_{k0}}{E_0^{(0)} - E_k^{(0)}} |\tilde{k}\rangle, \quad (20)$$

where  $F'_{k0} \equiv \langle k|F'|0\rangle$  are the matrix elements of  $F'$  in the basis of the unperturbed eigenkets, and  $E_l^{(0)} \equiv \hbar\hbar K \sqrt{P^2 - 4Q^2}$  is the  $l$ th unperturbed eigenvalue.

We need to transform the operators in (11) using the canonical transformation (12). After some algebra and neglecting the constant, we obtain

$$\begin{aligned} F' = & \hbar K [c^2 s^2 (\delta b^{\dagger 4} + \delta b^4) \\ & + 2(c^3 s + c s^3) (\delta b^{\dagger 3} \delta b + \delta b^{\dagger} \delta b^3) \\ & + (c^4 + s^4 + 4s^2 c^2) \delta b^{\dagger 2} \delta b^2] \\ & + 2\hbar K \alpha [(c s^2 + c^2 s) (\delta b^{\dagger 3} + \delta b^3) \\ & + (s^3 + c^3 + 2c^2 s + 2s^2 c) (\delta b^{\dagger 2} \delta b + \delta b^{\dagger} \delta b^2)], \end{aligned} \quad (21)$$

where we have made the definitions  $s \equiv \sinh r$  and  $c \equiv \cosh r$ . If we substitute (21) into (20) we obtain the following expression for the perturbed state

$$|\Phi^{(1)}\rangle = k_0 |\tilde{0}\rangle + k_3 |\tilde{3}\rangle + k_4 |\tilde{4}\rangle, \quad (22)$$

where  $k_0 = 1$ , and

$$k_3 = -\frac{2\sqrt{6} \hbar K \alpha}{3\sqrt{P^2 - 4Q^2}} (c s^2 + c^2 s), \quad (23)$$

$$k_4 = -\sqrt{3/2} \left( \frac{\hbar K}{\sqrt{P^2 - 4Q^2}} \right) c^2 s^2. \quad (24)$$

For the parameters chosen here, the largest of the coefficients  $|k_3|$  and  $|k_4|$  is about  $k_0/15$ . So, treating these terms as a small perturbation to  $|\tilde{0}\rangle$  seems a valid approach and, in fact, improves for larger values of  $\alpha$ .

The most direct way to see the effect of the non-Gaussian terms is to compare the Wigner function of

the perturbed state with that of the state obtained by assuming Gaussian statistics. We begin by calculating the symmetrically ordered characteristic function,  $\chi(\eta) = \text{Tr}\{\rho \exp(\eta \delta a^\dagger - \eta^* \delta a)\}$ , where we take the density matrix to be,  $\rho = |\Phi^{(1)}\rangle \langle \Phi^{(1)}|$ .

If we transform the operators in  $\chi(\eta)$  using (12), and then normally order all the operators taking care with their commutation relations, we arrive at

$$\chi(\eta) = \sum_{\{i,j\} \in \{0,3,4\}} k_i^* k_j \chi_{ij}, \quad (25)$$

where

$$\begin{aligned} \chi_{ij} = & \exp \left[ -\frac{1}{2} (\eta c - \eta^* s) (\eta^* c - \eta s) \right] \\ & \times \langle i | \exp(\delta b^\dagger (\eta c - \eta^* s)) \exp(-\delta b (\eta^* c - \eta s)) | j \rangle. \end{aligned} \quad (26)$$

The Wigner function of the state is then simply found by taking the two-dimensional inverse Fourier transform of this function. In practice, this is calculated numerically using a fast-Fourier-transform algorithm.

In Fig. 1, the one standard deviation contour of the Wigner function of this perturbed state is compared with the unperturbed state. A significant ‘‘bending’’ is evident. The state of constant intensity is shown for comparison and is seen to closely match the curvature of the condensate state. We conclude that the condensate is very nearly in a number squeezed state.

We would now like to study how this ‘‘banana’’ state evolves with time. In particular, we would like to consider the phase diffusion effects. We consider evolution while no measurements are being made. This means that there are none of the effects, associated with measurement, which try to localize the phase [8,9]. The Hamiltonian describing the time evolution of this system is

$$H = \hbar \tilde{\omega} a^\dagger a + \hbar K a^{\dagger 2} a^2, \quad (27)$$

which gives

$$a(t) = \exp[-i(\tilde{\omega} + 2Ka^\dagger a)t] a(0). \quad (28)$$

For some initial state,  $|i\rangle$ , we wish to investigate how  $\langle i|a(t)|i\rangle$  varies with time. The eigenstates of (27) are clearly the number states. We therefore write the initial condensate state as a superposition of number states,

$|i\rangle = \sum_n c_n |n\rangle$ , since the evolution of the condensate will be very simple in this representation. This gives,

$$\langle a(t) \rangle = \sum_n c_{n-1}^* c_n \sqrt{n} \exp[-2iK(n - \bar{N})t], \quad (29)$$

where  $\bar{N}$  is the mean number of atoms in the condensate and we have transformed to a frame rotating at frequency  $\mu/\hbar$  to eliminate the deterministic motion of the condensate in phase space.

The form of expression (29) is familiar from the Jaynes–Cummings model of quantum optics, which describes the interaction between a single-mode radiation field and a two-level atom, and exhibits the phenomenon of collapses and revivals [10]. The same behaviour is expected here [11]. We can see directly from Eq. (29) that  $\langle a(t) \rangle$  is periodic in time with period,  $T = \pi/K$ . This period is the time at which the phases of all the different components of the state get back in step with one another.

Collapse times may be estimated by looking at the spread of frequencies present in the wavepacket for particle numbers in the range,  $n = \bar{N} \pm \Delta n/2$ , where  $\Delta n$  is the standard deviation of the number distribution. From (29), it is clear that this frequency spread is  $\Delta\Omega = 2K\Delta n$ . The collapse time,  $t_{\text{coll}}$ , is estimated to be the time at which the spread in phase is  $2\pi$ , which gives,  $t_{\text{coll}} \approx \pi/K\Delta n = T/\Delta n$ . Since this depends on the number distribution, an experimental study of the timescale of the phase diffusion should reveal information about the number squeezing of the condensate. We consider three cases of initial states.

For a coherent state, the set of coefficients,  $\{c_n\}_{n=0}^{\infty}$ , is given by the familiar Poisson distribution with mean,  $\bar{N}$ . This will serve as a useful comparison to the result for the true condensate state. Secondly, we consider the case of the amplitude squeezed state  $|\alpha, r\rangle$  which was obtained when we assumed Gaussian statistics. The number distribution for this state is well known [12]. Finally, we consider the number squeezed state which corresponds to the true condensate state. To find the set of coefficients for this state, we simply fit a Gaussian in the number and phase quadratures to the perturbed state shown in Fig. 1.

By substituting these coefficients into (29), we can compare the rates of phase diffusion for the three cases. In Fig. 2, we have plotted the real part of (29) as a function of time. This variable is proportional to

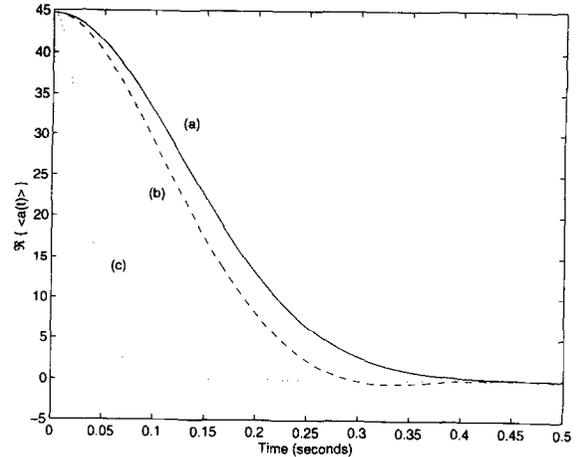


Fig. 2. Phase collapses for (a) the number squeezed condensate state, (b) the amplitude squeezed Gaussian state, and (c) a coherent state with the same mean number of atoms.

the mean X-quadrature value of the state, since  $a = \frac{1}{2}(X + iY)$ .

The timescales of the collapses are seen to depend on the statistics of the condensate. The more strongly number squeezed the state is, the longer that the condensate retains its phase. The timescales for the true condensate state and a coherent state differ by a factor of about 4.3. This significant difference means that an experimental investigation of the rate of phase diffusion should give clear evidence for the state of the condensate.

In practice, the diffusion of a condensate could be observed experimentally by studying the fringes formed by the interference of that condensate with a phase reference condensate [9]. As the phase becomes less well defined with time, we would expect the visibility of the fringes to decrease. A plot of the time dependence of this visibility should therefore reveal the timescale of the phase diffusion.

The theory developed here, however, is true only for a system on which no measurements are being made. Any measurement will, of course, alter the state of the condensate. For this reason, a new condensate must be identically prepared for each trial and then the visibility of the interference fringes measured after a different diffusion time. This will allow the timescale of the collapses to be determined and provides a possible scheme for experimentally testing the state of the condensate.

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