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Phase Standard for Bose-Einstein Condensates

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We discuss how a consistent phase standard for Bose-Einstein condensates may be defined. We show that it has the properties we would wish for in a phase standard: it is not corrupted by subsequent comparisons. A quantum jump technique is employed to study the time evolution of a three mode condensate system on which we make measurements, which entangle the modes and so establish relative phases between them. By establishing, in turn, the phases of two condensates relative to a reference condensate, we show that the relative phase between them can be predicted accurately. The existence of such a phase standard gives a precise definition to the phase of a condensate. [S0031-9007(99)08970-X]

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A central problem in the theory of matter wave sources is whether we can attribute a definite phase to a condensed gas. Can such a phase be defined in a manner that everyone will agree on? Is it more than a convenient fiction, or an unwarranted extrapolation of the ideas of spontaneous symmetry breaking from infinite systems to the finite condensates produced in the laboratory?

It can be argued that one can speak of the phase of a condensate only relative to another. Mølmer [1], in particular, has shown that isolated atomic fields cannot have nonvanishing mean amplitudes and the concept of an absolute phase for a condensate has no meaning. Such a conclusion gives special importance to the role of a “phase standard”: a reference condensate with which the phases of other condensates can be compared and the arbitrariness eliminated. Recent work [2,3] has demonstrated that, by making measurements which entangle two condensates, a relative phase can be generated between them even if they are initially in states of an undefined phase. The question arises whether one such condensate can act as a consistent phase reference, and so be assigned the role of phase standard.

This concept has been discussed in previous work both in the context of Josephson junctions [4,5] and atomic condensates [6]. The latter authors considered a scheme in which measurements induced a relative phase between

condensates, initially in number states with no initial phase information. They showed, in particular, that the interference patterns observed are indistinguishable from those obtained if the condensates start in coherent states, i.e., states of a well-defined phase. From this observation, they proposed that if relative phases were generated between each of two condensates (A and B) and a third one (C), the relative phase between A and B can be predicted from the results of the other two measurements. Such a proposal has C acting in precisely the role of a phase standard.

Leggett has shown that such a phase standard cannot exist if the system is allowed to reach equilibrium with the environment [4]. Although he suggests that a phase standard might be meaningful on a shorter time scale [7]. In this Letter, we test this proposal and include the key effect of measurements establishing the phase. We also investigate whether a phase is preserved in a measurement; in other words, if we repeat the measurement do we get the same result?

The entanglement between the three modes is crucial to such a scheme, and we need to keep a full record of its evolution. Previous methods which use conditional probabilities to determine the position (or time) of detection of the next atom in interference schemes [2,3] will not work. We need to keep track of the quantum state of the system at all times and hence follow a new calculational route.

We have chosen to apply the quantum jump method (see, for example, [8]) to a system of two condensates incident on a beam splitter to study the evolution of the system as atoms are detected. This enables us to keep track of the entanglement throughout a particular realization and may be viewed as a representative history of an experiment [9]. We will see that the transitive nature of the phase holds for any such realization. Of course, in this case we will not have to think of many realizations, since the idea of a phase standard is that it is a single, unique reference.

The setup is analogous to the theoretical work of Javanainen and Yoo [2], who analyzed the case of two initially spatially separated condensates that are allowed to overlap and the positions of atoms in the interference pattern recorded. An entanglement between the two “modes” is established due to the fact that one cannot know from which condensate the atoms have come. This entanglement leads to a relative phase seen in the buildup of an interference pattern in the spatially detected atoms. For convenience, we consider the time analog of this scheme, observing the time, rather than the position, at which atoms are detected. As in the spatial model, we would expect a relative phase to develop. This temporal scheme has the advantage of allowing us to consider condensates in trap ground states rather than in momentum eigenstates.

There have been various proposals for measuring the relative phase between condensates. These are based on recording interference patterns [2] or by inducing Raman transitions which couple the condensates [10,11]. The precise way in which the measurement is made does not matter as all of these schemes rely on detecting a superposition of the two fields. We have chosen to use a beam splitter to produce this superposition but a variant of the same analysis would work for any of these schemes.

We consider a 50:50 beam splitter with a condensate as the input to each of the two ports. The two condensate modes are identified with the operators a and b . If we transform to a frame rotating at the frequency of mode a , ω_a , the field operators at the two output ports of the beam splitter are

$$C_1 = \sqrt{\kappa/2}(a + ibe^{-i\Omega t}), \quad (1)$$

$$C_2 = \sqrt{\kappa/2}(ia + be^{-i\Omega t}), \quad (2)$$

where κ is the rate of detection of atoms and $\Omega = \omega_b - \omega_a$.

We wish to calculate the quantum state of the system conditioned on all the previous detections, $|\psi_c\rangle$. The procedure for simulating this measurement process is

(i) Determine the probability that an atom is detected at port i in a time interval, Δt ,

$$P_i(t, \Delta t) = \Delta t \langle \psi_c(t) | C_i^\dagger C_i | \psi_c(t) \rangle. \quad (3)$$

Then generate random numbers, r_i , from a flat distribution between zero and one, and compare with P_i .

(ii) If $r_i < P_i$, a detection is made in port i during Δt and the system jumps to the renormalized form,

$$|\psi_c(t)\rangle \rightarrow \frac{C_i |\psi_c(t)\rangle}{\sqrt{\langle \psi_c(t) | C_i^\dagger C_i | \psi_c(t) \rangle}}. \quad (4)$$

(iii) If $r_1 > P_1$ and $r_2 > P_2$, there is no detection in the interval, Δt , and the system evolves as

$$|\psi_c(t)\rangle \rightarrow \exp[-iH_{\text{eff}}\Delta t] |\psi_c(t)\rangle. \quad (5)$$

We have defined $\hbar \equiv 1$, and the effective Hamiltonian, H_{eff} , is given by $H_{\text{eff}} = H_0 - i\kappa(a^\dagger a + b^\dagger b)/2$, where H_0 is the system Hamiltonian. The state needs to be renormalized after each step since H_{eff} is non-Hermitian.

(iv) Repeat these steps to propagate the state in time until the desired number of atoms has been detected.

We can summarize this procedure as a stochastic Schrödinger equation (SSE), which the reader may find helpful. For each port, this equation contains a stochastic term which consists of the change in the state vector when a detection takes place at that port, multiplied by a random variable which determines whether or not an atom is detected at that time. There is also a deterministic term describing the evolution when an atom is not detected. More formally, we can write the SSE as

$$d|\psi_c\rangle = \left\{ dN_1 \left(\frac{C_1}{\sqrt{Q_c^1(t)}} - 1 \right) + dN_2 \left(\frac{C_2}{\sqrt{Q_c^2(t)}} - 1 \right) - dt \left[iH_0 + \frac{\kappa}{2} (a^\dagger a + b^\dagger b - \langle a^\dagger a \rangle - \langle b^\dagger b \rangle) \right] \right\} |\psi_c\rangle, \quad (6)$$

where we have defined $Q_c^i(t) \equiv \langle \psi_c(t) | C_i^\dagger C_i | \psi_c(t) \rangle$. The random variables, dN_1 and dN_2 , must have the properties,

$$dN_i(t)^2 = dN_i(t), \quad (7)$$

$$E[dN_i(t)] = Q_c^i(t) dt = P_i(t, dt), \quad (8)$$

where $E[\cdot]$ denotes an ensemble average.

Another way to consider (6) is as an “unraveling” of the master equation [8,12–14]. We can show this by calculating the time derivative of the system’s density matrix, $\rho(t) = |\psi_c(t)\rangle\langle\psi_c(t)|$, on an ensemble average. A simple calculation using (6) gives the master equation for the system, which is what we would expect for an ensemble average over all the trajectories. There is no unique unraveling of the master equation. However, Eq. (6) is the special case for which each trajectory can be thought of in terms of a representative experimental run [9]. This is what we want, since when comparing a condensate with a phase standard, we must consider a single trajectory not an ensemble average.

For an initial number state, the master equation can lead only to mixtures of number states and, as such, does not allow for the buildup of phase. However, this need not concern us since, as we shall see in the results

below, each trajectory demonstrates the same fixed and predictable phase relationship between the modes. Our concept of a phase standard is that there is only one, and so this must obey the same phase relationship as that of a single trajectory. The phase information would be wiped out only if we were to take the ensemble average over all phase standards. The notion of a phase standard thus relies on everybody using the same reference condensate.

We are now in a position to study how the phase builds up between condensates when we detect atoms at the output ports of the beam splitter. We use the calculational procedure outlined in Eqs. (3)–(5). The three mode setup is shown in Fig. 1. We take each condensate initially to be in a number state with 1000 atoms, and so the system begins with no phase information. The total system will remain in a state of a known number throughout the simulation since we detect and record every atom that escapes from the traps. As we shall see, however, this does not prevent relative phases from developing between the three parts of the system.

For simplicity, we set the trap frequencies and detection rate so that $\omega_a = \omega_c = \omega_b/4 = 40\kappa$. In the rotating frame, we also set the system Hamiltonian, H_0 , equal to zero. This last condition amounts to considering atoms which do not interact with one another. It is very straightforward to generalize H_0 to include interactions between atoms, which give rise to interesting effects such as collapses and revivals of the phase [15].

To begin with, we allow atoms to leak out of traps a and b and we record the times at which atoms are detected

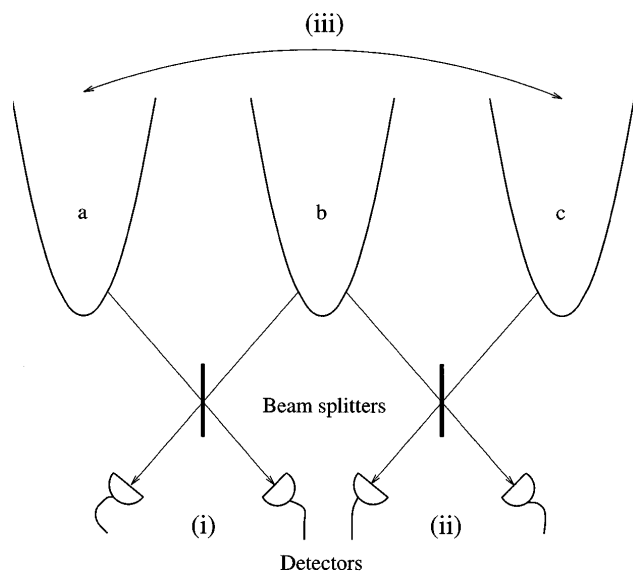


FIG. 1. The three mode setup. (i) In the first stage, condensates a and b are the inputs to a beam splitter and we record atomic detection times at the two output ports. (ii) In the second stage we repeat the process but with b (now entangled with a) and c as the two inputs. (iii) Finally, we measure the relative phase between a and c and compare with the results of the phase measurements in (i) and (ii).

in beam splitter ports 1 and 2. We continue until about 10% of the atoms in traps a and b (i.e., about 200) have been detected. Throughout the simulation, we dynamically calculate the relative phase between a and b , ϕ_{ab} . This is given by

$$\phi_{ab}(t) = \arg\{\langle \psi_c(t) | a^\dagger b | \psi_c(t) \rangle\}. \quad (9)$$

The measurement scheme serves the dual role of both creating and measuring the relative phase. This phase (9) may be found by taking the difference in the number of atoms detected at each port, $D(t)$, per time interval, Δt . For zero phase difference, we would expect equal numbers of detections at each port, and for a leading (lagging) b by $\pi/2$ we would expect more detections at port 1(2). This means that for degenerate modes, we would expect to see a sinusoidal time dependence of the difference in numbers of atoms detected at the two ports. In analogy with the position of the spatial fringes in interference experiments, we can use the “time position” of the temporal fringes as a measurement of the relative phase.

On average, the difference in the number of atoms detected at each port per time interval is given by

$$D(t)/\Delta t = \langle \psi_c(t) | C_1^\dagger C_1 - C_2^\dagger C_2 | \psi_c(t) \rangle \quad (10)$$

$$= i\kappa(\langle a^\dagger b \rangle e^{i\Omega t} - \langle b^\dagger a \rangle e^{-i\Omega t}). \quad (11)$$

We can write

$$\langle a^\dagger b \rangle = |\langle a^\dagger b \rangle| e^{i\phi_{ab}(t)}, \quad (12)$$

where $\phi_{ab}(t)$ contains the nondeterministic part of the relative phase due to the randomness of each trajectory. With this substitution, we get

$$D(t)/\Delta t = -2\kappa \sin[\Omega t + \phi_{ab}(t)]. \quad (13)$$

So the relative phase is given by the argument of the sinusoidal plot of the difference in the number of atoms detected at each port per time interval. For our results, we subtract the known deterministic component, Ωt .

A plot of ϕ_{ab} against time is shown in Fig. 2(a). As expected, the relative phase is initially undefined. Then, as atoms are detected, it fluctuates for a while before settling down to a fixed constant value, Φ_{ab} . This value is random and varies for measurements made on identically prepared systems. At the end of this detection process, modes a and b are entangled and mode c is unaffected. The state vector for the system is given by

$$|\psi_c\rangle = \left(\sum_{i=N-l}^N c_i |2N-l-i\rangle_a |i\rangle_b \right) |N\rangle_c, \quad (14)$$

where N is the initial number of atoms in each trap, l is the number of atoms detected ($l < N$), and $\{c_i\}$ are the coefficients determined by the numerical simulation.

In the second part of the simulation, we take mode b (now entangled with a) as the input to one port of a beam splitter, and mode c as the other input. As before, we record the times that atoms are detected at the two output ports until roughly 10% of the atoms have been detected.

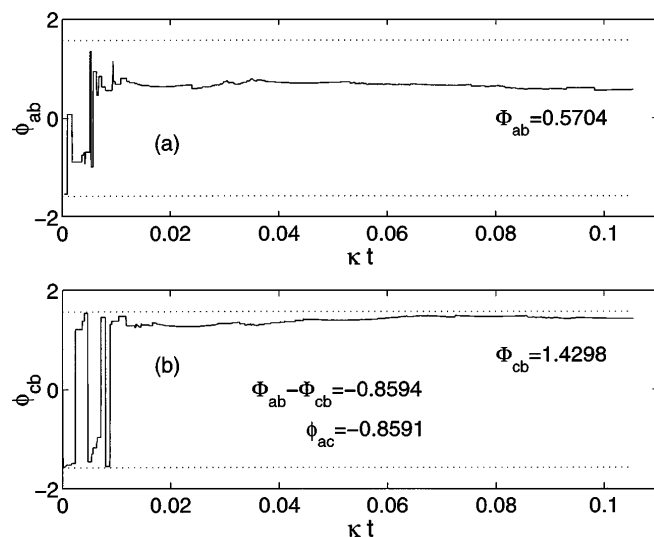


FIG. 2. Plot of the establishment of a relative phase between the condensate modes. The time evolution of the relative phase between modes a and b is shown in (a) and between b (now entangled with a) and c in (b). In each case, the relative phase [modulo(π)] is plotted against the dimensionless quantity κt , where κ is the rate of detection of atoms.

A plot of the relative phase between modes c and b , ϕ_{cb} , is shown in Fig. 2(b). As in the first part, this eventually settles down to a fixed constant value, Φ_{cb} , and this varies randomly for measurements made on identically prepared systems.

An important observation is that the entanglement of b and c does not affect the existing entanglement between a and b , so long as the modes are not too severely depleted in the process. This is a very desirable property for a phase standard since it means that the standard is not corrupted in the process of comparing it with another condensate.

In this simulation, we consider that there is no independent loss from the traps. Such a process, which is currently under investigation, will corrupt the entanglements and so degrade the phase standard. This explains why the scheme will not work if the system is allowed to reach equilibrium with the environment [4]. We investigate the system on a different time scale: long before it reaches equilibrium. For our system, this can be long enough for useful experiments. The effect of interactions and loss are treated by Sinatra and Castin [16].

At the end of this second measurement stage, modes a and c are each entangled with b . The state vector for the system is

$$|\psi_c\rangle = \sum_{j=N-m}^N \sum_{i=2N-m-j}^N d_{i,j} |3N - i - j - m\rangle_a |j\rangle_b |i\rangle_c, \quad (15)$$

where m is the total number of atoms detected in both measurement stages ($m < N$), and $\{d_{i,j}\}$ are the coefficients determined by the numerical simulation.

Finally, in the third part, we calculate the relative phase between a and c , ϕ_{ac} . This is given by

$$\begin{aligned} \phi_{ac} &= \arg[\langle \psi_c | a^\dagger c | \psi_c \rangle] \\ &= \arg \left\{ \sum_{j=N-m}^N \sum_{i=2N-m-j}^N d_{i,j} d_{(i-1),j}^* \right. \\ &\quad \left. \times \sqrt{i(3N - i - j - m + 1)} \right\}. \quad (16) \end{aligned}$$

Proposals have been made for how this could be measured [10]. Alternatively, we could perform another interference measurement. The difference here is that the relative phase exists *a priori* and is not built up by the measurement. For this scheme to work successfully as a phase standard, we require that $\phi_{ac} = \Phi_{ab} - \Phi_{cb}$. From the simulation shown in Fig. 2, we see that there is remarkable agreement. In fact, many simulations were performed and in all of them we were able to predict the relative phase between a and c with great accuracy.

Once the phases have been established between the modes, further measurements give the same results. The relative phases are now fixed and are encoded in the entanglements and further phase measurements do not corrupt these.

Of course, we could carry on and measure the relative phase between a fourth condensate and our phase standard. We know that this measurement will not destroy the entanglements with the other condensates, so we could then predict the relative phases between this condensate and the other two. The only limitation on how many times this phase reference can be used is the number of atoms it contains. Provided that the initial number of atoms is very large, many measurements can be made before the standard is depleted. As our simulations show, only a small fraction of the atoms need to be removed to make a measurement, which means our sample is not destroyed in the process. Such an arrangement demonstrates all the properties that we require in a phase standard.

The existence of a phase standard for Bose-Einstein condensates is of considerable importance. It gives us a clear and precise definition of the phase of a condensate. We can now speak sensibly of a condensate's phase, interpreting this as being relative to a phase standard.

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