Precision measurement with an optical Josephson junction

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(Received 26 January 2007; published 8 June 2007)

We present a theoretical study of a type of Josephson device, the so-called "optical Josephson junction" [Y. Shin *et al.* Phys. Rev. Lett. **95**, 170402 (2005).]. In this device, two condensates are optically coupled through a waveguide by a pair of Bragg beams. This optical Josephson junction differs from the usual Josephson junction where condensates are weakly coupled by tunneling through a barrier. We discuss the use of this optical Josephson junction, for making precision measurements.

DOI: 10.1103/PhysRevA.75.063607 PACS number(s): 03.75.Lm, 03.75.Gg, 03.75.−b

I. INTRODUCTION

Atom optics has undergone rapid development since the advent of Bose-Einstein condensates (BECs) of atomic gases [1]. We have seen the realization of atom lasers, matter-wave solitons [2], and interferometry using coherent matter waves [3]. In addition, new techniques of manipulating cold atoms have been devised such as Bragg scattering [4] and magnetic waveguides [5], etc. These sophisticated techniques may lead to the realization of quantum limited measurement [6] and applications in quantum information processing [7].

Recently, Saba *et al.* [8] and Shin *et al.* [9] have demonstrated the Josephson effect in atom optics. Two beams of atoms are optically extracted from separate trapped BECs using Bragg scattering [4]. The two atomic beams subsequently overlap and interfere, and measurement of the interference pattern creates a relative phase between the BECs they originate from. This pioneering experiment showed that Josephson coupling of these two spatially separate systems can be made through an intermediate "transport" system. It is quite different from conventional Josephson devices, such as superconducting systems [10], and Bose-Einstein condensates [11], in which the two condensates involved are connected by tunneling and their wave functions have a small direct overlap.

In the "optical Josephson junction" (OJJ), two trapped condensates are connected by out-coupling small fractions of the condensates using Bragg scattering [4]. Out-coupled atoms move along a magnetic waveguide [5] before being transferred to the partner condensate [9]. The size of the overall Josephson coupling can now be controlled by varying the strength of the Bragg beams. The phase of the coupling phase between these two condensates can also be tuned by adjusting the phase shifts of the out-coupled atoms [9]. This control gives the OJJ significant advantages over the conventional junction. The OJJ could be used, for example, to implement the precision scheme proposed by Dunningham and Burnett (DB) [6] in which they consider Heisenberg limited measurement (i.e., the measurement uncertainty scales as $1/N$, where N is the number of atoms) using an entangled BEC trapped in a double-well potential.

We shall, therefore, study the use of the OJJ in the implementation of the scheme suggested by Dunningham *et al.* [6]. This scheme could, in principle, be used to make precision measurements of atomic scattering strengths or the acceleration due to gravity. In this proposed experimental scheme, the initial measurement of phase is performed by first pulsing the Josephson coupling. The final phase information produced is encoded in atom number fluctuations. These fluctuations can be determined from the collapse and revival of the relative phase between the condensates. This scheme provides a way to measure the interaction induced phase produced with Heisenberg limited accuracy.

The paper is organized as follows: In Sec. II, we introduce our theoretical model of the OJJ. In Sec. III, we derive an effective Hamiltonian under the two-mode approximation for the trapped BECs. In Sec. IV, we study the implementation of the DB scheme in this system. For completeness we have given a brief review of the theory of the Bragg scattering process in the Appendix.

II. BASIC MODEL

We consider two trapped condensates coupled into a 1D ring-form waveguide using Mth-order Bragg scattering [8,9] as shown in Fig. 1. This Bragg scattering coupling is envisaged to be the $2M$th-order multiphoton Raman process

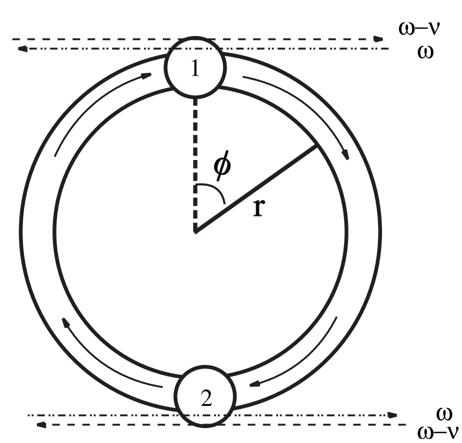

FIG. 1. The schematic of the optical coupling of two trapped condensates 1 and 2, and the out-coupling atoms are transferred via a waveguide in the ring form with radius r. The pair of Bragg beams with frequencies ω and ω-ν are denoted by a dashed-dotted line and dashed line, respectively.

[4,12]. The necessary stimulated emission and absorption is driven by the two counterpropagating Bragg beams acting as pump and probe field with a controlled frequency difference. The wavelength and wave number of the Bragg beam with frequency ω are λ and $k=2\pi/\lambda$, respectively.

A small portion of the BECs are outcoupled from the two trapped BECs with definite momenta $\hbar k_1$ and $\hbar k_2$ at the Mth-order Raman process. For simplicity, we assume the two momenta $\hbar k_1$ and $\hbar k_2$ have similar magnitude but opposite sign, for $\hbar^2 k_{1,2}^2/2m = M\hbar\nu$ and m is the mass of an atom. In our model Bragg beams can produce "recapture" of atoms to the condensate and should enable us to see Josephson effects. In contrast, in the experiments [8,9] the atoms were strongly coupled out and not transferred back to the trapped portion of the condensates [9].

We now discuss the Hamiltonian of this system and derive an effective two-mode Hamiltonian to describe the optical Josephson effect. The Hamiltonian of the total system has the form

$$H = H_0 + H_{\text{ring}} + H_{\text{couple}}, \tag{1}$$

where H_0, H_{ring}, and H_{couple} are the Hamiltonians of BECs 1 and 2, the out-coupled atoms in the one-dimensional (1D) ring, and the coupling between the trapped atoms (1 and 2) and the out-coupled atoms in the ring, respectively. The Hamiltonian for the trapped atoms has the form

$$H_0 = \sum_{j=1}^{2} \oint ds \left[\Psi_j^\dagger(s) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial s^2} + V_j(s) \right) \Psi_j(s) + \frac{U_0}{2} \Psi_j^\dagger(s)\Psi_j^\dagger(s)\Psi_j(s)\Psi_j(s) \right]. \tag{2}$$

Here, $\Psi_j(s)$ and $V_j(s)$ are the field operator and trapping potential, respectively, for condensate j, U_0 is the interaction strength and $j=1,2$, and $s=r\phi$ for r is the radius of the ring and ϕ is the polar angle of the ring (see Fig. 1).

We consider that the two condensates are held in traps such that the single-mode approximation is valid [15]. The field operator $\Psi_j(s)$ can then be approximated by $c_j\psi_j(s)$, where c_j and $\psi_j(s)$ are, respectively, the annihilation operator and mode function of the jth trap and $j=1,2$. If we take the system to be symmetric so that $\Psi_1(s)$ has the same form as $\Psi_2(s)$, the Hamiltonian H_0 can be written as

$$H_0 = E_0(c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{\hbar\kappa}{2}[(c_1^\dagger c_1)^2 + (c_2^\dagger c_2)^2], \tag{3}$$

where

$$E_0 = \oint ds \psi_j^*(s) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial s^2} + V_j(s) \right) \psi_j(s), \tag{4}$$

is the eigenenergy of the two modes and

$$\kappa = U_0 \oint ds |\psi_j^*(s)\psi_j(s)|^2 \tag{5}$$

is the self-interaction strength.

We should note that atoms can be out-coupled from a trapped condensate with near unit efficiency using first-order Bragg scattering [4]. The efficiency of higher order Bragg scattering is, however, significantly lower [4]. This will lead to a modification to our effective coupling both in phase and amplitude. In the subsequent discussion, we shall consider the case of low-order Bragg scattering with high efficiency. The Hamiltonian representing the effective coupling between the trapped atoms and free atoms can then be written, in the form

$$H_{\text{couple}} = \gamma \oint ds [\Psi_f^\dagger(s)\Psi_1(s) + e^{-i\pi\delta k r}\Psi_f^\dagger(s)\Psi_2(s)] + \text{H.c.}, \tag{6}$$

where γ is the coupling between condensate in the trap BECs and waveguide [13,14], $\delta k = k_1 + k_2 - 4k$ [9], respectively. The relative phase shift $\pi\delta k r$ is generated during the flight between the two condensates. The phase shift δk directly depends on the Bragg beams and the momenta $k_{1,2}$ which can be adjusted in the experiment [9].

The Hamiltonian of the 1D ring has the form:

$$H_{\text{ring}} = \oint ds \left[\Psi_f^\dagger(s) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial s^2} \right) \Psi_f(s) + \frac{U_0}{2}\Psi_f^\dagger(s)\Psi_f^\dagger(s)\Psi_f(s)\Psi_f(s) \right], \tag{7}$$

where $\Psi_f(s)$ is the field operator of the condensate in this ring-form waveguide. The condensate is free to move in the angular direction, but confined radially. We can justify this approximation by considering the magnitude of the kinetic energy and the strength of the nonlinear interaction. The kinetic energy $\hbar\omega_{k_j} = \hbar^2 k_j^2/2m$ of the outcoupled condensates by the Bragg beams is with $\omega_{k_j} \sim 10^2$ kHz [9] whereas the mean-field interaction shift of atoms in the waveguide will be at the most tens of Hz. From this we can argue that the out-coupled condensates can be treated as freely evolving when in the ring waveguide.

The operator superposition that represents atoms in the ring is thus given by

$$\Psi_f(s) = (2\pi r)^{-1/2}(e^{ik_1 s}g_{k_1} + e^{ik_2 s}g_{k_2}). \tag{8}$$

Here $2\pi r$ is the circumference of the loop, $k_{1,2}$ should satisfy the boundary condition $k_{1,2}r = \alpha_{1,2}$, where $\alpha_{1,2}$ is an integer, and g_{k_j} are the annihilation operators for the ground states with the momentum $\hbar k_j$. The Hamiltonian for the atoms in the ring can be written as

$$H_{\text{ring}} = \sum_{j=1}^{2} \hbar\omega_{k_j} g_{k_j}^\dagger g_{k_j}. \tag{9}$$

III. EFFECTIVE HAMILTONIAN

We now derive an effective Hamiltonian to describe the Josephson effect in this system. To do this we adiabatically eliminate the nearly free intermediate states through which the two traps are coupled. In this way we find the effective coupling Hamiltonian in the form

$$H_{\mathrm{couple}} = \hbar \sum_{j=1}^{2} \gamma'(k_j)(c_1^\dagger + e^{i\theta}c_2^\dagger)g_{k_j} + \mathrm{H.c.} \tag{10}$$

Here $\gamma'(k_j)=\gamma(2\pi r)^{-1/2}\oint ds e^{-ik_j s}\psi_j(s)$, $j=1,2$, and $\theta=\pi\delta kr$. We assume that the two coupling strengths $\gamma'(k_j)$ are approximately equal, i.e., $\oint ds e^{-ik_1 s}\psi_1(s) \approx \oint ds e^{-ik_2 s}\psi_2(s)$. The Heisenberg equation of motion of g_{k_j} then has the form

$$i\dot{g}_{k_j} = \omega_{k_j} g_{k_j} + \gamma'(k_j)(c_1 + e^{-i\theta}c_2). \tag{11}$$

We use the adiabatic approximation, i.e., $\dot{g}_{k_j} \approx 0$ as the condition $\omega_{k_j} \gg |\gamma'(k_j)|$ is, we assume, satisfied,

$$g_{k_j} \approx -\frac{\gamma'(k_j)}{\omega_{k_j}}(c_1 + e^{-i\theta}c_2). \tag{12}$$

The validity of this adiabatic approximation relies on the flight time between the two trapped condensates being short compared to the period of the Josephson oscillation. We can estimate this flight time at approximately 0.1 to 1 ms, with the velocity of the out-coupled atoms ($\sim$10 m ms^{-1}), and the length of the ring ($\sim$1 to 10 μm) [9]. Thus, the Josephson frequency must be much lower than 1 to 10 kHz which can easily be achieved by adjusting the strength of the Bragg beams [12].

The resulting effective two-state Hamiltonian can thus be written

$$H_{\mathrm{eff}} = \left(E_0 - \frac{\hbar\Omega}{2}\right)(c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{\hbar\kappa}{2}\left[(c_1^\dagger c_1)^2 + (c_2^\dagger c_2)^2\right] - \frac{\hbar\Omega}{2}(e^{-i\theta}c_1^\dagger c_2 + e^{i\theta}c_2^\dagger c_1), \tag{13}$$

where $\Omega = 2\sum_{j=1}^{2}\gamma'^2(k_j)/\omega_{k_j}$. It is noteworthy that this effective two-mode Hamiltonian is akin to the Josephson Hamiltonian of the external [15] and the internal [16] BEC systems. This Josephson coupling can be controlled by using the strength of the Bragg beam to vary g. The phase of Josephson coupling can be adjusted by the phase shift θ that is proportional to δk and the arclength between the BECs. This is a most useful feature to have in the use of the OJJ in applications. It is convenient to write the effective Hamiltonian in terms of angular momentum operators thus

$$J_x = \frac{1}{2}(c_1^\dagger c_2 + c_2^\dagger c_1), \tag{14}$$

$$J_y = \frac{1}{2i}(c_1^\dagger c_2 - c_2^\dagger c_1), \tag{15}$$

$$J_z = \frac{1}{2}(c_1^\dagger c_1 - c_2^\dagger c_2), \tag{16}$$

where the total atom numbers $N = c_2^\dagger c_2 + c_1^\dagger c_1$ is conserved. The Hamiltonian for the system can then be written in the final form

$$H_{\mathrm{eff}} = \hbar\kappa J_z^2 - \hbar\Omega(\cos\theta J_x + \sin\theta J_y) + C, \tag{17}$$

where $C = (E_0 - \hbar\Omega/2)N + \hbar\kappa N^2/4$. This constant C does not affect the quantum dynamics of the system, and can be ignored.

IV. IMPLEMENTATION OF PRECISION MEASUREMENT

In this section, we discuss the implementation of precision measurement on the optical Josephson junction. Dunningham and Burnett proposed a precision measurement scheme using a number-squeezed BEC trapped in a double-well potential [6]. This scheme requires the active control of the Josephson coupling. The phase information can be finally obtained by measuring the number fluctuations detected via the visibility of the interference fringes of the two condensates. There are some serious limitations of the BEC double-well system with respect to the control of the Josephson coupling. The Josephson coupling strength depends exponentially on the height of the potential barrier between the two wells. This means that the height of the potential barrier must be controlled very accurately indeed in order to obtain the correct coupling. In addition, when we want to make the Josephson coupling large, there would necessarily need to be a large spatial overlap between the wave functions in two wells. If this were so, the two-mode approximation would no longer be valid. This problem can now be avoided by using the OJJ and it is, therefore a promising route to implementing this scheme.

We shall now describe how one could realize explicitly the DB scheme. For simplicity, we choose the phase shift $\theta = \pi$, so that the effective Hamiltonian has the form

$$H_{\mathrm{eff}} = \hbar\kappa J_z^2 + \hbar\Omega J_x. \tag{18}$$

We first prepare the initial state as a number-squeezed state $|N/2\rangle_1|N/2\rangle_2$ which contains a definite number of atoms trapped in two wells. This can be prepared by adiabatically switching off the Josephson coupling strength so that the condensates are isolated in two different traps in the Fock regime, i.e., $\kappa \gg \Omega N$ [17]. For simplicity, we consider only the perfectly squeezed case here. This can be difficult to achieve in practice, particularly for large N. However, as long as some degree of squeezing can be achieved, it should still be possible to perform measurements with an accuracy that surpasses the shot-noise limit [18].

We can use this system to measure a relative phase between the condensates as follows. A large Josephson coupling is turned on rapidly with the strength $\Omega \gg \kappa N$ by using Bragg pulses with $\Omega = \Omega^*$. To describe the situation when the coupling in on, it is convenient to write the state in the new eigenbasis, i.e., the symmetric mode $\alpha = (c_1 + c_2)/\sqrt{2}$ and antisymmetric mode $\beta = (c_1 - c_2)/\sqrt{2}$ with respect to two traps. The quantum state in this new basis, has the form [6]

$$|\psi\rangle = \sum_{m=0}^{N/2} (-1)^m C_m |2m\rangle_\alpha |N-2m\rangle_\beta, \tag{19}$$

where $C_m = \sqrt{(2m)!(N-2m)!}/[2^{N/2}m!(N/2-m)!]$. It is worth noting that this superposition of states is relatively robust against the particle loss as noted in [19]. In this regime, there is a small energy difference, Ω^*, between the symmetric and antisymmetric modes, different phases result for the terms in the superposition. Then, the system is held for a certain time $t=t^*$ to allow the natural evolution of the system.

Next, the Josephson coupling is suddenly switched off fast with respect to coupling between wells, but slow with respect to the inverse energy level spacing in each well [22]. Thus, the state is conveniently expressed in terms of the number basis in each well. The quantum state becomes [6]

$$|\psi'(0)\rangle = \frac{1}{2^{N/2}(N/2)!}\sum_{n=0}^{N}(-1)^n D_n|n\rangle_1|N-n\rangle_2, \tag{20}$$

where

$$D_n = \sum_{p=\max\{0,n-N/2\}}^{n/2}\binom{N/2}{p^*} \times \binom{p^*}{p}\sqrt{n!(N-n)!}(i\sin\phi)^{p^*}(2\cos\phi)^{n-2p} \tag{21}$$

for $p^*=N/2-n+2p$ and $\phi=\Omega^* t^*$. This completes the measurement of the phase ϕ which is recorded in the quantum state now. The information of this phase ϕ can be obtained from the number uncertainty Δn, where $n=c_1^\dagger c_1=N-c_2^\dagger c_2$ is the number of atoms in the trap 1 (or 2). From Eq. (20), the number uncertainty can be calculated as [6]

$$\Delta n = \frac{N}{2\sqrt{2}}\sin\phi. \tag{22}$$

This number uncertainty is of order N. According to the number-phase uncertainty relation $\Delta n\Delta\phi\sim 1$, we can see that the uncertainty of the phase ϕ, for the minimum uncertainty state, is of order N^{-1}, i.e., it scales with N in the same manner as the Heisenberg limit.

This number variance can be experimentally determined from the interference pattern. Bragg scattering provides a convenient method to determine the relative phase. A small fraction of the atoms are coupled out horizontally from these two condensates and allowed to interfere with each other. The relative phase of the two trapped condensates can be determined from the interference pattern of these two overlapping waves [20,23]. The interference pattern [24], $I=\int dx[\Psi_{f1}^\dagger(x)\Psi_{f2}(x)+\Psi_{f2}^\dagger(x)\Psi_{f1}(x)]$, is directly proportional to the interference terms of two trapped condensates, where Ψ_{f1} and Ψ_{f2} are the field operator of two out-coupled condensates from the jth trap, and x is the coordinate in the horizontal direction. Following a similar treatment to the preceding section and taking $\theta=\pi$, the intensity of the interference fringes, I, of these two out-coupled condensates can be obtained as [24]

$$I = \frac{\gamma'(k_1)\gamma'(k_2)}{\omega_{k1}\omega_{k_2}}\langle\psi'(\tau)|c_1^\dagger c_2 + c_2^\dagger c_1|\psi'(\tau)\rangle, \tag{23}$$

where τ is the time of holding the system with the merely nonlinear interaction of the atoms. This state vector is given by

$$|\psi'(\tau)\rangle = \frac{1}{2^{N/2}(N/2)!}\sum_{n=0}^{N}(-1)^n e^{-i\kappa[n^2+(N-n)^2]\tau/2}D_n|n\rangle_1|N-n\rangle_2. \tag{24}$$

Thus, the intensity, I, is proportional to

$$I(\tau) \propto \sum_n D_{n+1}^* D_n\sqrt{(n+1)(N-n)}e^{i\kappa(2n+1-N)\tau} + D_{n-1}^* D_n\sqrt{n(N-n+1)}e^{i\kappa(N-2n+1)\tau}. \tag{25}$$

The collapse times t_{coll} can be estimated by considering the particle numbers in the range, $n=N/2\pm\Delta n/2$. Hence, the collapse times of the relative phase t_{coll} is about $\pi/\kappa\Delta n$ [20,23]. This collapse time can be determined by holding the system with the nonlinear interaction as a function of time τ and measuring the corresponding intensity I with different τ's. We can therefore determine the number fluctuation and hence the required phase information.

The experimentally observable collapse time can be short and the effects of the decay and decoherence effects relatively modest. This means that an accurate measurement of the number variance should be possible using this method. Although the revival time can reveal the phase information, it takes a much longer time to observe. Keep in mind that the measurement of collapse and revival time of a Bose-Einstein condensate has been demonstrated in the experiment [21].

V. CONCLUSION

In this paper, we have presented a study of a microscopic model of the "optical Josephson junction" and, derived the effective Hamiltonian for the operation of this device. We have also discussed how this system can be used to implement Heisenberg limited precision measurement, through the examination of the collapse time of an interference pattern.

We noted above that the lower efficiency of higher order Bragg scattering may well limit the effective coupling between the two BECs. Future experiments will be needed to show to what extent this can be avoided. On the other hand, it is very interesting to compare this optical-based Josephson junction with its solid-state counterpart.

ACKNOWLEDGMENTS

One of the authors (H.T.N.) was supported by the Croucher Foundation. One of the authors (K.B.) thanks the Royal Society and Wolfson Foundation for support. One of the authors (J.D.) was funded by the EPSRC.

APPENDIX: BRAGG SCATTERING

In this appendix, the basic mechanism of the condensates using the Bragg scattering is briefly reviewed. The pump-probe mechanism has been discussed in detail in Ref. [12]. The pump and probe fields impart a momentum $2k$ to the ground state of the condensates each time by coupling to the excited state with a large detuning $\Delta=\tilde{\omega}-\omega$ between the two-level atoms with the energy difference $\hbar\tilde{\omega}$ and the laser field. To elucidate this process, we study the first-order Bragg resonance case by considering a time-independent Hamiltonian in the interaction picture and assume the zero ground-state energy for g_0 which is given by

$$H_{1st} = \hbar\Delta_1 e_k^\dagger e_k + \hbar(\Delta_1 - \Delta_2) g_{2k}^\dagger g_{2k} + \hbar\lambda(g_0^\dagger e_k + e_k^\dagger g_0) + \hbar\lambda'(g_{2k}^\dagger e_k + e_k^\dagger g_{2k}), \quad \text{(A1)}$$

where e_{nk} and g_{nk} are the annihilation operators for the excited and ground states with the momentum nk; λ and λ' are the coupling strength of the pump field and the probe field of this pair of Bragg beams; and $\Delta_1 = \Delta + \omega_k$ and $\Delta_2 = \Delta + \omega_k - \omega_{2k} + \nu$ are the detuning between the ground and the excited states with the different momenta and the Bragg beams, for $\omega_{nk} = \hbar(nk)^2/2m$.

The equations of motion for the different momentum states are given by

$$i\dot{g}_0 = \lambda e_k, \quad \text{(A2)}$$

$$i\dot{e}_k = \Delta_1 e_k + \lambda g_0 + \lambda' g_{2k}, \quad \text{(A3)}$$

$$i\dot{g}_{2k} = (\Delta_1 - \Delta_2) g_{2k} + \lambda' e_k. \quad \text{(A4)}$$

At the Bragg resonance, the detuning $\Delta_1 - \Delta_2 = 4\omega_k - \nu$ equals zero at $\nu = 4\omega_k$. The excited state e_k can be adiabatically eliminated as $\Delta_1 \gg \lambda, \lambda'$, i.e.,

$$e_k = -\frac{1}{\Delta_1}(\lambda g_0 + \lambda' g_{2k}). \quad \text{(A5)}$$

Therefore, the equations of motion for these two different ground states have the form

$$i\dot{g}_0 = -\frac{1}{\Delta_1}(\lambda^2 g_0 + \lambda\lambda' g_{2k}), \quad \text{(A6)}$$

$$i\dot{g}_{2k} = -\frac{1}{\Delta_1}(\lambda\lambda' g_0 + \lambda'^2 g_{2k}). \quad \text{(A7)}$$

Clearly, we can see that these two momentum states are effectively coupled with each other at the first-order Bragg resonance.

In general, we can consider the Mth-order Bragg scattering which is a $2M$th multiphoton Raman process. Within this process, the different momentum modes are virtually excited but they can be adiabatically eliminated because of energy conservation being unfavorable. It is legitimate to consider the effective coupling between the trapped condensates and the free momentum states $\hbar\omega_{k_{1,2}} = M\nu$ at the Bragg resonance only in which the energy is conserved. The explicit form of the effective coupling between the trapped and free states, γ, has been found as [12]

$$\gamma = \left| \frac{(\lambda\lambda'/\Delta)^M}{[(M-1)!]^2 \omega_{2k}^{M-1}} \right|. \quad \text{(A8)}$$

The detailed analysis can be found in Ref. [12].

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