Precision measurement scheme using a quantum interferometer

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We present a scheme that makes use of quantum mechanical entanglement to significantly enhance the measurement precision accessible by interferometers. This is achieved by making use of a special type of interferometer composed of two "quantum beam splitters," which each transform the input particles into a macroscopic superposition at the outputs. We show that this technique not only enables phase measurements to be made with Heisenberg limited precision, but also overcomes the major practical problems of detector inefficiencies and imperfect beam splitters. It may provide a promising route to implementing sub-shot-noise limited measurement schemes in the laboratory.

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The ability to make increasingly precise measurements of physical quantities has long been an important challenge in physics. One of the key developments in the field of optical measurements was the interferometer, which enabled path length differences to be detected through phase shifts with unprecedented accuracy. Further advances in metrology were proposed with the discovery of nonclassical (e.g., squeezed) states of light [1–5]. These included a proposal for enhancing the precision that could be achieved in an interferometer by using light with reduced phase fluctuations as the input [6,7]. Since then, considerable effort has been devoted to the detection of phase shifts below the "shot noise limit," where the phase uncertainty scales as, $\Delta\theta \sim 1/\sqrt{N}$ and N is the total number of particles involved [8]. Here we demonstrate a promising route for achieving this in the laboratory that is remarkably immune to the effects of imperfections in the particle detectors. This may be of importance in a number of areas of physics including the detection of gravitational waves and in a range of quantum information schemes.

Standard interferometers are limited by shot noise, which results from using a stream of uncorrelated particles to make a measurement. It has, however, been argued that by making use of "cooperative" effects between the particles, i.e., quantum mechanical entanglement, it should be possible to reach the Heisenberg limit, where the measurement accuracy scales inversely with N.

In 1993, Holland and Burnett demonstrated how the Heisenberg limit could be achieved in an interferometer by using dual Fock states at the input, i.e., when each input port of the interferometer has precisely the same number of particles [9]. This paper stimulated considerable interest in such a scheme, particularly with regard to its detailed implementation. Kim *et al.* showed that the number correlated light from an optical parametric oscillator (OPO) or amplifier (OPA) would be a practical alternative to the dual Fock state input, by analyzing the effects of decorrelation and the statistics of the input photon pairs [10]. In these schemes, however, the phase information cannot be determined simply by measuring the population imbalance at the output ports, as is the case for standard interferometry. Instead we need to measure coincidences or correlations between the particles [10]. Unfortunately, such measurements are extremely sensitive to any deviation from unit detector efficiency, which suggests that these schemes are likely to be impractical [11]. Dunningham, Burnett, and Barnett demonstrated how this substantial problem can be overcome with an atomic analogue of this scheme, which involves disentangling the atomic state before measurements are made on it [12]. Another promising scheme which overcomes the problem of detector efficiencies involves measuring the collapses and revivals of the visibility of interference fringes for Bose-Einstein condensates [13]. In the optical regime, recent experiments have provided evidence for Heisenberg-limited interferometry with ultrastable twin beams [14].

In this paper, we wish to explore a different route to achieving sub-shot-noise limited measurements that involves creating maximally entangled states inside the interferometer. These so-called NOON states [15] have the form,

$$|\psi\rangle = \frac{1}{\sqrt{2}}[i|N,0\rangle + |0,N\rangle], \tag{1}$$

where $|k,l\rangle$ represents the number of particles on each of the two paths. The NOON state is a macroscopic superposition of all the particles being on one path of the interferometer and all on the other. Bollinger *et al.* proposed the idea of using such maximally entangled states to make precise measurements of the frequency of atomic transitions [16]. They showed that the resolution that could be achieved by this technique scales inversely with the total number of particles.

Various proposals have been made for producing NOON states in the laboratory. These include the use of Fredkin gates [17–20], quantum switching [17,21], and coupling a quantum superposition state to a beam splitter [22]. Experiments have successfully created NOON states with three photons [23], and four $^9Be^+$ ions [24], and could in principle be scaled up to larger numbers. A particularly promising theoretical proposal involves the making use of ordinary beam splitters and nonlinear unitary evolution to produce states of

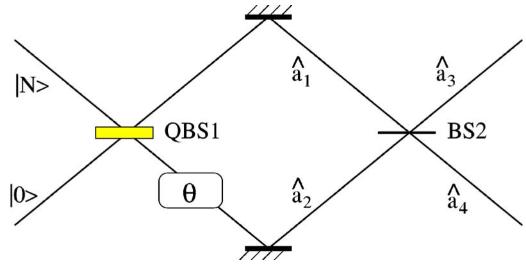

FIG. 1. (Color online) A schematic of a Mach-Zehnder interferometer with one quantum beam splitter (QBS1) and an ordinary 50:50 beam splitter (BS2). The quantum beam splitter has the property of creating a superposition of the form of (1) with all the particles in one output port and all in the other. There is a phase shift, θ, on one arm that we wish to measure.

the form of (1) [25]. The latter type of evolution could be implemented with nonlinear crystals for photons or by exploiting the interactions between atoms in Bose-Einstein condensates. We shall, in any case, show that we do not need to create perfect NOON states to realize much of the associated advantage in interferometry.

We shall use the general term "quantum beam splitter" to denote any scheme or device that creates a NOON state from particles that are not initially entangled. The special property of these devices is that if we feed particles into one input port and a vacuum into the other, the output is a superposition of all the particles at one port and all the particles at the other. In contrast, if we passed a stream of uncorrelated particles through an ordinary 50:50 beam splitter, we would obtain a binomial distribution of particles at the outputs. The operation of a quantum beam splitter can be seen to be equivalent to that of an ordinary beam splitter if all the particles were somehow "stuck together." Throughout this paper, we shall treat the quantum beam splitter as a "black box" to preserve the generality of the scheme and to avoid unnecessary details. However, as outlined above, it should be noted that there are realistic practical schemes for implementing these devices.

The two terms of (1) will acquire different phases depending on the length of the path they follow inside the interferometer. Other authors have made suggestions for how this phase could be read out, which include passing the state through an ordinary 50:50 beam splitter and making intensity correlation measurements on the outputs. Here we show that these read-out schemes impose prohibitively strict conditions on the detector efficiencies when N is large. We then show how we can overcome this problem by proposing a "quantum interferometer" composed of two quantum beam splitters. An analysis of this scheme shows that it should enable us to achieve Heisenberg-limited resolution for phase measurements and depends only weakly on the detector efficiencies. This suggests a promising route for making sub-shot-noise measurements in the laboratory.

We begin by considering a Mach-Zehnder interferometer fed with a number state $|N\rangle$ at one input port and a vacuum state $|0\rangle$ at the other (see Fig. 1). The first element (QBS1) in this interferometer is a quantum beam splitter. This acts on the input state to give the maximally entangled NOON state (1). The path length difference between the two arms of the interferometer gives rise to a phase factor $e^{-iN\theta}$ between the terms of the superposition and the state incident on the second beam splitter is

$$|\psi\rangle_{\mathrm{II}} = \frac{1}{\sqrt{2}}[i|N,0\rangle + e^{-iN\theta}|0,N\rangle]. \tag{2}$$

In this scheme, the second beam splitter (BS2) is simply an ordinary 50:50 one.

We now come to the crucial limitation of the setup depicted in Fig. 1, which is that one has to measure a sufficiently high order correlation function to see the effects of the phase shift. These correlation functions have the general form,

$$\langle :\hat{n}^r: \rangle = \langle :\hat{n}(x_1)\hat{n}(x_2)\cdots\hat{n}(x_r): \rangle, \tag{3}$$

where $\hat{n}$ is the photon number operator, r is the order of the correlation function, and $\{x_i\}$ are the positions of each measurement [15,26,27].

Let us suppose that we measure the rth order correlation function at one of the two outputs of BS2. If we label the two outputs 3 and 4, then the annihilation operators corresponding to these modes can be written as

$$\hat{a}_3 = \frac{1}{\sqrt{2}}(i\hat{a}_1 + \hat{a}_2), \tag{4}$$

$$\hat{a}_4 = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2). \tag{5}$$

This enables us to write the correlation function for mode 3 as $\langle :\hat{n}_3^r: \rangle = {}_{\mathrm{II}}\langle\psi|(\hat{a}_3^\dagger)^r(\hat{a}_3)^r|\psi\rangle_{\mathrm{II}}$, where $|\psi\rangle_{\mathrm{II}}$ denotes the state just before BS2. When we substitute (2) into this expression, all but four terms vanish: $\langle\hat{a}_1^{\dagger r}\hat{a}_1^r\rangle$, $\langle\hat{a}_2^{\dagger r}\hat{a}_2^r\rangle$, $\langle\hat{a}_1^{\dagger r}\hat{a}_2^r\rangle$, $\langle\hat{a}_2^{\dagger r}\hat{a}_1^r\rangle$. For the case of $r<N$, the correlation function $\langle :\hat{n}_3^r: \rangle$ is completely independent of θ and so there are no interference fringes. For the case $r=N$, we get

$$\langle :\hat{n}_3^r: \rangle = \frac{N!}{2^N}[1-(-1)^{N/2}\sin N\theta] \quad \text{for even } N, \tag{6}$$

$$\langle :\hat{n}_3^r: \rangle = \frac{N!}{2^N}[1+(-1)^{(N+1)/2}\cos N\theta] \quad \text{for odd } N. \tag{7}$$

A similar calculation holds for $\langle :\hat{n}_4^r: \rangle$. The fact that the order of correlation must be the same as the number of input particles to see any effect of the phase shift means that we must detect every particle. This imposes severe constraints on the scheme. In particular, the efficiency, μ, of the detectors must satisfy $\mu > 1-1/N$, which renders the scheme impractical for any more than a few particles.

We would now like to consider how we can overcome this serious impediment by considering the case of a Mach-Zehnder interferometer with two quantum beam splitters (see Fig. 2). In this case, the output from the second quantum beam splitter (QBS2) with input (2) is

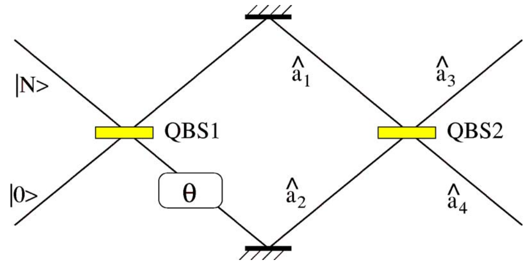

FIG. 2. (Color online) A schematic of a Mach-Zehnder quantum interferometer composed of two quantum beam splitters (QBS1 and QBS2). By feeding N particles into one input and 0 into the other and then dectecting the particles at the two outputs (corresponding to the annihilation operators $\hat{a}_3$ and $\hat{a}_4$), Heisenberg limited measurements can be made of the phase shift, θ.

$$|\psi\rangle_{\mathrm{III}} = \frac{1}{2}[(-1+e^{-iN\theta})|N,0\rangle + i(1+e^{-iN\theta})|0,N\rangle]. \quad (8)$$

The mean number of particles detected at output ports 3 and 4 are now given by

$$_{\mathrm{III}}\langle\psi|\hat{n}_3|\psi\rangle_{\mathrm{III}} = N(1-\cos N\theta)/2 \quad (9)$$

and

$$_{\mathrm{III}}\langle\psi|\hat{n}_4|\psi\rangle_{\mathrm{III}} = N(1+\cos N\theta)/2, \quad (10)$$

respectively, and the number variance at each port is $(\Delta n_3)^2 = (\Delta n_4)^2 = N^2(\sin^2 N\theta)/4$. We should note that, for each measurement, we detect all N particles at one port or all N at the other. It is the relative probability of these two events occuring that enables us to measure the phase, θ. A calculation of the square of the phase uncertainty for detectors with perfect efficiency gives

$$(\Delta\theta)^2 = \frac{(\Delta n_3)^2}{\left(\frac{\partial\langle\hat{n}_3\rangle_{\mathrm{III}}}{\partial\theta}\right)^2} = \frac{1}{N^2}, \quad (11)$$

which shows that this scheme allows us to reach the Heisenberg limit.

We would now like to consider how imperfect detectors do not prevent us from getting close to the Heisenberg limit. Let us suppose that we measure the phase shift θ with a detector with efficiency μ, where $0 \leqslant \mu \leqslant 1$. According to the model of nonideal photodetection, the detected field mode is described by a photon annihilation operator, $\hat{a}'_3 = \sqrt{\mu}\hat{a}_3 + \sqrt{1-\mu}\hat{v}_3$, where $\hat{v}_3$ is the annihilation operator for the vacuum state mode [28,29]. The number operator for the detected photons is then given by, $\hat{n}'_3 = \hat{a}'^{\dagger}_3\hat{a}'_3$. This allows us to obtain the relations, $\langle\hat{n}'_3\rangle_{\mathrm{III}} = \mu\langle\hat{n}_3\rangle_{\mathrm{III}}$ and $(\Delta n'_3)^2 = \mu^2(\Delta n_3)^2 + \mu(1-\mu)\langle\hat{n}_3\rangle_{\mathrm{III}}$. Using these relations we can obtain an expression for the phase uncertainty,

$$(\Delta\theta)^2 = \frac{(\Delta n'_3)^2}{\left(\frac{\partial\langle\hat{n}'_3\rangle_{\mathrm{III}}}{\partial\theta}\right)^2} = \frac{1}{N^2} + \left(\frac{1-\mu}{\mu}\right)\frac{1}{N^3\cos^2(N\theta/2)}. \quad (12)$$

The first term on the right-hand side, which is independent of the detector efficiency, represents the ideal case. The second term accounts for detector imperfections and hence vanishes for perfect detectors, $\mu=1$. The key point is that this second term scales as $1/N^3$, for values of $N\theta$ not too close to $(1+2p)\pi$, where p is an integer. This means that the destructive effects of realistic detectors are negligible for NOON states with large N. This is a remarkable result as detector efficiencies are a major obstacle to beating the standard quantum limit in other precision measurement schemes. Moreover, for large N, the signal-to-noise ratio (SNR), is given by

$$\mathrm{SNR} = \frac{\langle\hat{n}'_3\rangle}{\Delta n'_3} = |\tan(N\theta/2)|, \quad (13)$$

which means that we should obtain a clear signal when $N\theta$ lies in the interval $(\pi/2+2p\pi,\ 3\pi/2+2p\pi)$, and p is an integer.

An indication of the robustness of this scheme to imperfections in the beam splitting process can be obtained by considering the more general case where QBS2 splits the incoming photons into a superposition of $N-m$ and m $(0 \leqslant m \leqslant N/2)$ photons at the outputs [25]. In this case, the output state from the interferometer is given by,

$$|\psi\rangle_{\mathrm{III}} = \frac{1}{2}\sum_{m=0}^{N/2} C_m[(-1+e^{-iN\theta})|N-m,m\rangle + i(1+e^{-iN\theta})|m,N-m\rangle], \quad (14)$$

where $\sum_m |C_m|^2 = 1$. Following a similar calculation to above, the square of the phase uncertainty, $(\Delta\theta)_0^2$, is given by

$$(\Delta\theta)_0^2 = \frac{1}{N^2} + \frac{4(\Delta m)^2}{[N(N-2\bar{m})\sin N\theta]^2}, \quad (15)$$

where $(\Delta m)^2 \equiv \bar{m^2} - \bar{m}^2$, $\bar{m} \equiv \sum_m |C_m|^2 m$, and $\bar{m^2} \equiv \sum_m |C_m|^2 m^2$. In the case that $\bar{m} \ll N/2$ and $\bar{m^2} \ll N^2/4$, Eq. (15) can be written as

$$(\Delta\theta)_0^2 \approx \frac{1}{N^2} + \frac{4(\Delta m)^2}{N^4 \sin^2 N\theta}. \quad (16)$$

This reduces to $(\Delta\theta)_0^2 \approx 1/N^2$ so long as $N\theta$ is not too close to an integer multiple of π. We can, therefore, still expect to approach the Heisenberg limit in this more general case. Similar results hold when both beam splitters are imperfect. This is because, if the first quantum beam splitter is slightly imperfect, the state inside the interferometer is very close to a NOON state and should allow measurements resolutions close to the Heisenberg limit. Since the output beam splitter only marginally degrades the signal, it should still be possible to obtain enhanced measurement resolution when both beam splitters are imperfect.

Finally, we are interested in how the sensitivity of the phase shift measurement is affected by finite detector efficiencies in this case of imperfect beam splitting. A straightforward calculation for the phase uncertainty yields, $(\Delta\theta)^2 = (\Delta\theta)_0^2 + (\Delta\theta)_1^2$, where $(\Delta\theta)_0^2$ is given by (15) and $(\Delta\theta)_1^2$ is given by

$$(\Delta\theta)_1^2 = \left(\frac{1-\mu}{\mu}\right)\frac{2[N-(N-2\bar{m})\cos N\theta]}{[N(N-2\bar{m})\sin N\theta]^2}. \qquad (17)$$

If we consider the same case as before, i.e., $\bar{m} \ll N/2$ and $\bar{m}^2 \ll N^2/4$, the overall phase uncertainty can be written as

$$(\Delta\theta)^2 \approx \frac{1}{N^2} + \left(\frac{1-\mu}{\mu}\right)\left[\frac{N+2\bar{m}}{N^4\cos^2(N\theta/2)} + \frac{4\bar{m}}{N^4\sin^2 N\theta}\right], \qquad (18)$$

which reduces to (12) in the limit $\bar{m} \to 0$. So long as $N\theta$ is not too close to an integer multiple of π, the term in square brackets of expression (18) scales as $1/N^3$. As before, this correction due to detector imperfections is negligible with respect to the first term for large N. This means that we can achieve Heisenberg limited resolution in the measurement of the phase shift even if QBS2 does not create a perfect superposition of all the photons going one way with all of them going the other. This suggests that this scheme may work even if the beam splitter creates an incoherent mixture of states, so long as each of these states is not too different from NOON states. Such a property is clearly important with regard to the experimental practicalities of the scheme. It is also important that a full treatment accounts for decoherence of the particles in the interferometer. Such an effect rapidly destroys the superposition in NOON states and needs to be carefully avoided. There is also the possibility of loss of particles during the quantum beam splitting process. This has been shown to be equivalent to imperfect beam splitting and so, as discussed above, does not destroy the measurement. A thorough treatment of the effects of decoherence will feature in forthcoming work.

We should finish by emphasizing that we believe that the implementation of quantum interferometers should be feasible in upcoming experiments. Experiments have already created NOON states for small numbers of photons and ions and these schemes should be able to be scaled up to larger numbers. This, combined with the fact that they are robust to imperfections both in the beam splitters and detectors, means they may well provide a valuable route for achieving precision measurements beyond the shot-noise limit.

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