Fourier multiport devices

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A formalism for multiport devices is developed in which the creation and annihilation operators at the output are related with those at the input through a finite Fourier transform. By considering two specific inputs of number states and coherent states, we show how these devices can be used to produce multimode Schrödinger cat states and can be combined to create multipath interferometers. We introduce uncertainty relations for the distributions of particles in the input modes and the correlations between the output modes. These allow us to determine how the phase resolution of the output state scales with the number of ports and particles. This scaling is of fundamental interest for schemes that seek to use multipath interferometers for enhanced measurement precision.

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I. INTRODUCTION

Devices that manipulate the entanglements between quantum systems are of great interest from both a theoretical and a practical point of view. Mach-Zehnder interferometers, for example, have played an important role in quantum optics [1–9] and have far-reaching potential applications that include optical gyroscopes and gravitational wave detectors. More general multiport devices have also been studied in the literature for both photons and atoms [10–12]. These devices perform unitary transformations and could be used as gates in quantum-information processing schemes on networks or as generalized beam splitters in interferometers. The interest in these devices has been increasing rapidly as progress is made toward their physical implementation. Multiport beam splitters with six or eight ports have been experimentally demonstrated for photons [13] and experiments with Bose-Einstein condensates trapped in optical lattices suggest an exciting avenue for their realization with atoms [14,15].

In this paper we consider a special case of $2d$-port devices in which the creation and annihilation operators $\{a_M^\dagger, a_M\}$ of photons (or atoms) at the d inputs are related to the creation and annihilation operators $\{A_M^\dagger, A_M\}$ of photons (or atoms) at the d outputs through a finite Fourier transform. This can be thought of as the multiport analog of a beam splitter. In this case, each output is a combination of all the inputs with equal weights, which leads to interesting results. We develop a theoretical formalism for the operation of this device and consider the results for particular inputs. We show that there are uncertainty relations that connect the distribution of photons in the various modes at the input, with the correlations between the various modes at the output. These uncertainty relations are constraints which provide a deeper insight into the system.

The format of this paper is as follows. In Sec. II we begin by presenting the basic formalism and the unitary transformation associated with Fourier multiport devices. In Sec. III we use this formalism to study particular examples which elucidate the operation of the device. In particular, we show that a certain superposition of number states at the input produces a multimode Schrödinger cat state at the output comprised from number states with all photons in one mode. We also consider the case of coherent state inputs and show that a coherent state in one of the inputs (with vacua in all other inputs) will split into d coherent states with an equal average number of photons at each of the outputs. Furthermore, the phases of the coherent states at the output indicate the particular channel at which the coherent state entered the device. This could have useful applications in quantum optical networks and in the general area of multipath interferometry.

In Sec. IV A we consider the number distribution $\langle n_M\rangle$ (where $n_M = a_M^\dagger a_M$) of photons in the input modes; and the first-order correlation $\langle C_K\rangle$ (where $C_K = d^{-1/2}\Sigma_\Lambda A_\Lambda^\dagger A_{\Lambda-K}$) between the output modes. We show that there is an uncertainty relation which states that when one of these distributions is narrow, the other one is wide. Uncertainty relations involving higher moments of these quantities can also be formulated. For example, in Sec. IV B we consider the distribution Δn_M^2 which describes the uncertainty in the photon number in the input modes; and the distribution ΔC_K^2 which describes second-order correlations at the output modes. As before, we show that there is an uncertainty relation which inversely relates the widths of these distributions.

In Sec. V we use our results to analyze the phase uncertainties at the output ports. This enables us to derive a scaling law for the phase resolution in terms of the number of ports and the total number of particles involved. This result is of great importance for optimizing the measurement resolution that will be able to be achieved in proposed multiport interferometers. Finally, in Sec. VI we conclude with a discussion and an overview of our results.

II. FOURIER MULTIPORT DEVICES

We consider a $2d$-port device with d inputs and d outputs. We denote as $\{a, a^\dagger\}$ the annihilation and creation operators of photons at the input; and we use the notation

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$$a_M = \mathbf{1} \otimes \cdots \otimes a \otimes \cdots \otimes \mathbf{1},$$

$$a_M^\dagger = \mathbf{1} \otimes \cdots \otimes a^\dagger \otimes \cdots \otimes \mathbf{1},$$

$$[a_M, a_K^\dagger] = \delta(M,K) \tag{1}$$

for the annihilation and creation operators of photons at the Mth input. For convenience, the indices are integers modulo d; they belong in Z_d (the set of integers modulo d). $\delta(M,K)$ is Kronecker's delta [it is equal to 1 when $M=K$ (modulo d)]. In a similar way, we use the notation A_M and $A_M^\dagger$ for the annihilation and creation operators of photons at the output.

We consider multiport devices for which the annihilation and creation operators at the output are related to those at the input through the finite Fourier transform

$$A_M = U a_M U^\dagger = d^{-1/2} \sum_K a_K \omega^{KM}, \quad \omega = \exp\left(i \frac{2\pi}{d}\right),$$

$$A_M^\dagger = U a_M^\dagger U^\dagger = d^{-1/2} \sum_K a_K^\dagger \omega^{-KM}, \tag{2}$$

where U is the unitary transformation performed by the device. U is a special case of symplectic $\mathrm{Sp}(2n,R)$ transformations and is given explicitly below. The inverse relation is

$$a_M = d^{-1/2} \sum_K A_K \omega^{-KM}, \quad a_M^\dagger = d^{-1/2} \sum_K A_K^\dagger \omega^{KM}. \tag{3}$$

It is easily seen that the vacuum state $|0,\ldots,0\rangle$ is the same with respect to the $\{a_M\}$ operators, as it is with respect to the $\{A_M\}$ operators. But the number eigenstates of $n_K = a_K^\dagger a_K$ are different from the number eigenstates of $N_K = U a_K^\dagger a_K U^\dagger = A_K^\dagger A_K$; and we use the notation $|M_1,\ldots,M_d\rangle_a$ and $|M_1,\ldots,M_d\rangle_A$ for those states correspondingly:

$$n_K |M_1,\ldots,M_d\rangle_a = M_K |M_1,\ldots,M_d\rangle_a,$$

$$N_K |M_1,\ldots,M_d\rangle_A = M_K |M_1,\ldots,M_d\rangle_A,$$

$$|M_1,\ldots,M_d\rangle_A = U |M_1,\ldots,M_d\rangle_a. \tag{4}$$

We also use the terms "a photons" and "A photons" in relation to $\{a_K, a_K^\dagger\}$ and $\{A_K, A_K^\dagger\}$, respectively. We next use Eq. (2) and the relation

$$\frac{1}{d} \sum_M \omega^{M(K-\Lambda)} = \delta(K,\Lambda) \tag{5}$$

to prove that

$$\sum_M n_M = \sum_K N_K. \tag{6}$$

Therefore the Fourier multiport device preserves the total number of photons.

A. The unitary transformation

Having defined the Fourier multiport transformation, we would now like to find an explicit form for the operator U appearing in Eq. (2). We consider the operator

$$U = \exp\left[i \sum_{M,K} a_M^\dagger \Phi_{MK} a_K\right], \tag{7}$$

where Φ_{MK} is element (M,K) of a Hermitian $d \times d$ matrix Φ. It is known (e.g., [16]) that

$$U a_M U^\dagger = \sum_K F_{MK} a_K, \quad F = \exp(-i\Phi). \tag{8}$$

F is a $d \times d$ matrix and it is the matrix exponential of $-i\Phi$.

In our case we want F to be the finite Fourier matrix (e.g., [17,18])

$$F_{MK} = d^{-1/2} \omega^{MK}, \quad \omega = \exp\left(i \frac{2\pi}{d}\right). \tag{9}$$

This matrix obeys the relations

$$F F^\dagger = \mathbf{1}, \quad F^4 = \mathbf{1}. \tag{10}$$

The matrix Φ is simply the logarithm of the matrix F, times i:

$$\Phi = i \ln F. \tag{11}$$

For numerical calculations, the logarithm of a matrix is readily available in computer libraries (e.g., MATLAB).

For more theoretical work, we will express the matrix Φ in terms of its eigenvalues and eigenvectors. Equation (10) shows that the eigenvalues of F are $i, -1, -i, 1$. There exists a lot of literature on the eigenvalues and eigenvectors of F [17], and for our purposes it suffices to write F as

$$F = i\varpi_{(i)} - \varpi_{(-1)} - i\varpi_{(-i)} + \varpi_{(1)}, \tag{12}$$

where $\varpi_{(i)}$, $\varpi_{(-1)}$, $\varpi_{(-i)}$, $\varpi_{(1)}$, are projection operators into the eigenspaces corresponding to the eigenvalues $i, -1, -i, 1$, respectively. It is now easily seen that the matrix Φ can be written as

$$\Phi = -\frac{\pi}{2}\varpi_{(i)} - \pi\varpi_{(-1)} - \frac{3\pi}{2}\varpi_{(-i)}. \tag{13}$$

The $\varpi_{(1)}$ does not appear because it is multiplied by ln 1 and so vanishes. Therefore the operator U can be written as

$$U = U_{(i)} U_{(-1)} U_{(-i)},$$

$$U_{(i)} = \exp\left[-i\frac{\pi}{2} \sum_{M,K} a_M^\dagger \varpi_{(i)MK} a_K\right], \quad U_{(i)}^4 = \mathbf{1},$$

$$U_{(-1)} = \exp\left[-i\pi \sum_{M,K} a_M^\dagger \varpi_{(-1)MK} a_K\right], \quad U_{(-1)}^2 = \mathbf{1},$$

$$U_{(-i)} = \exp\left[-i\frac{3\pi}{2} \sum_{M,K} a_M^\dagger \varpi_{(-i)MK} a_K\right], \quad U_{(-i)}^4 = \mathbf{1}. \tag{14}$$

We have taken into account here that $\Sigma_{M,K} a_M^\dagger \varpi_{(i)MK} a_K$, $\Sigma_{M,K} a_M^\dagger \varpi_{(-1)MK} a_K$, $\Sigma_{M,K} a_M^\dagger \varpi_{(-i)MK} a_K$, and $\Sigma_{M,K} a_M^\dagger \varpi_{(1)MK} a_K$ commute with each other. It follows that

$$U^4 = \mathbf{1}. \tag{15}$$

III. EXAMPLES

In this section we discuss two specific examples of input states that elucidate the operation of the device. In the first, we consider how number states are transformed and show how superpositions of number states can be used to create Schrödinger cat states at the output. In the second, we consider coherent states and study some of their properties when they are passed through the device.

A. Number states

A number state with ν photons in the Mth mode can be written as

$$|0, \ldots, \nu^{(M)}, \ldots, 0\rangle_A = \frac{(A_M^\dagger)^\nu}{(\nu!)^{1/2}}|0, \ldots, 0\rangle, \tag{16}$$

where the superscript (M) indicates that ν is in the Mth mode. We then use Eq. (2) to prove

$$\omega^{KM}(A_M^\dagger)^\nu = d^{-\nu/2} \sum_{\nu_1,\ldots,\nu_d} \frac{\nu!}{\nu_1! \ldots \nu_d!} \times (a_1^\dagger)^{\nu_1} \ldots (a_d^\dagger)^{\nu_d} \omega^{-M(\nu_1+2\nu_2+\cdots+d\nu_d-K)} \tag{17}$$

where the sum is over all non-negative integers such that $\nu = \nu_1 + \cdots + \nu_d$. Summation over M, taking into account Eq. (5), gives

$$\omega^K(A_1^\dagger)^\nu + \cdots + \omega^{Kd}(A_d^\dagger)^\nu = d^{1-\nu/2} \sum_{\nu_1,\ldots,\nu_d} \frac{\nu!}{\nu_1! \cdots \nu_d!} (a_1^\dagger)^{\nu_1} \cdots (a_d^\dagger)^{\nu_d}, \tag{18}$$

where the sum is over all non-negative integers such that

$$\nu = \nu_1 + \cdots + \nu_d; \quad \nu_1 + 2\nu_2 + \cdots + d\nu_d = K(\text{mod } d). \tag{19}$$

The second constraint means that the number of terms in the sum of Eq. (18) is drastically reduced in comparison to Eq. (17) (it is divided by d). Acting with Eq. (18) on the vacuum we prove

$$|s_K\rangle = d^{-1/2}[\omega^K|\nu, \ldots, 0\rangle_A + \cdots + \omega^{Kd}|0, \ldots, \nu\rangle_A] = d^{(1-\nu)/2} \sum_{\nu_1,\ldots,\nu_d} \left[\frac{\nu!}{\nu_1! \cdots \nu_d!}\right]^{1/2} |\nu_1, \ldots, \nu_d\rangle_a, \tag{20}$$

where the summation is over non-negative integers which obey the constraints of Eq. (19). The left-hand side is a Schrödinger cat; it is a superposition of number states of A photons with ν photons in one of the modes and zero photons in the rest of the modes. The right-hand side is a superposition of number states of a photons. We can show that with respect to both A photons and a photons these states have the same average number of photons in each mode:

$$\langle \hat{N}_1 \rangle = \cdots = \langle \hat{N}_d \rangle = \frac{\nu}{d}, \quad \langle \hat{n}_1 \rangle = \cdots = \langle \hat{n}_d \rangle = \frac{\nu}{d}. \tag{21}$$

We can also show that the various states $|s_K\rangle$ are orthogonal to each other,

$$\langle s_K|s_\Lambda\rangle = \delta(K,\Lambda). \tag{22}$$

B. Coherent states

We now consider the case of a coherent state in the Mth mode. This can be written as

$$|0, \ldots, \beta^{(M)}, \ldots, 0\rangle_a = D_{a_M}(\beta)|0, \ldots, 0\rangle \tag{23}$$

where $D_{a_M}(\beta)$ is the displacement operator

$$D_{a_M}(\beta) \equiv \exp[\beta a_M^\dagger - \beta^* a_M] \tag{24}$$

and β is a complex number. The index a_M indicates that this is a displacement operator with respect to a_M and $a_M^\dagger$.

Using Eq. (2) it is straightforward to show that

$$\beta A_M^\dagger - \beta^* A_M = d^{-1/2}\sum_K [\beta\omega^{-KM}a_K^\dagger - \beta^* \omega^{KM}a_K] \tag{25}$$

and this leads to the relation

$$D_{A_M}(\beta) = D_{a_1}(d^{-1/2}\beta\omega^{-M})D_{a_2}(d^{-1/2}\beta\omega^{-2M}) \cdots D_{a_d}(d^{-1/2}\beta\omega^{-dM}). \tag{26}$$

Acting with these operators on the vacuum we get the coherent states

$$|0, \ldots, \beta^{(M)}, \ldots, 0\rangle_A = |d^{-1/2}\beta\omega^{-M}, \ldots, d^{-1/2}\beta\omega^{-dM}\rangle_a. \tag{27}$$

The left-hand side is a coherent state with respect to the A operators. It is a tensor product of vacua with the coherent state $|\beta\rangle$ in the Mth mode. The right-hand side is a coherent state with respect to a operators.

It is seen that coherent states with phases $2\pi M/d$ at the inputs produce a coherent state in one of the outputs and vacua in all other outputs. This is a multimode generalization of the well-known two-mode result. An interesting application of these states is in multimode interferometry schemes. If we were to start with a coherent state in one of the input modes, M, and vacua in all other inputs, we would get coherent states in all outputs with phases $2\pi M/d$. If we then applied a linearly varying phase across these output modes, the change in phase of mode j would be $\Delta\Theta_j = j\eta$. For $\eta = 2\pi/d$, the state becomes

$$|d^{-1/2}\beta\omega^{-(M-1)}, \ldots, d^{-1/2}\beta\omega^{-d(M-1)}\rangle_A \tag{28}$$

and if we transform it back through the Fourier device, we get

$$|0, \ldots, \beta^{(M-1)}, \ldots, 0\rangle_a, \tag{29}$$

i.e., the coherent state has moved along one mode relative to the initial state. This scheme is a multimode generalization of an interferometer. However, instead of measuring the phase shift via the population at two output modes, we measure it at d modes. This gives a finer measurement scale and may lead to enhanced precision.

Another possible application of this device is to split a coherent state into many other coherent states. It can be seen from Eq. (27) directly that a coherent state with an average number of $|\beta|^2$ photons will be split into d coherent states each with an average number of $|\beta|^2/d$ photons.

Equation (27) can be generalized into

$$|\beta_1, \ldots, \beta_d\rangle_A = |B_1, \ldots, B_d\rangle_a, \quad B_M = \sum_K (F^{-1})_{MK}\beta_K, \tag{30}$$

where F^{-1} is the inverse Fourier matrix

$$(F^{-1})_{MK} = d^{-1/2}\omega^{-MK}. \tag{31}$$

We note that if the d-dimensional vector $\{\beta_K\}$ is an eigenvector of the Fourier matrix F^{-1} [17,18], then $B_M = \lambda\beta_M$ where λ is the corresponding eigenvalue, which takes one of the values 1, i, -1, $-i$. We see that for the eigenvectors with eigenvalue 1, the corresponding coherent states exit unaltered from the device; and for the eigenvectors with eigenvalue i, -1, $-i$, the phase of each of the coherent states changes by $\pi/2$, π, $3\pi/2$, correspondingly.

IV. UNCERTAINTY RELATIONS

We can gain a deeper understanding of these systems by considering the uncertainty relations between the input and output modes. In the following subsections we focus on two particular cases and discuss uncertainty relations involving first- and second-order correlations of the output modes.

A. First-order correlations

We use Eq. (3) in conjunction with Eq. (5) to prove that

$$n_M \equiv a_M^\dagger a_M = d^{-1/2}\sum_K C_K\omega^{KM}, \tag{32}$$

where

$$C_K = d^{-1/2}\sum_\Lambda A_\Lambda^\dagger A_{\Lambda-K},$$

$$[C_K, C_M] = 0, \quad C_K^\dagger = C_{-K}. \tag{33}$$

It is seen that the operators n_M and C_K are related through a finite Fourier transform. Taking the expectation values with respect to an arbitrary state $|s\rangle$ we find

$$\langle n_M\rangle = d^{-1/2}\sum_K \langle C_K\rangle\omega^{KM}. \tag{34}$$

$\langle n_M\rangle$ is the distribution of the number of a photons in the various modes, in the state $|s\rangle$. $\langle C_K\rangle$ is the A-photon first-order correlation between the modes Λ and $\Lambda-K$, summed over all Λ. $\langle C_0\rangle$ in particular is the total average number of photons in the state $|s\rangle$ (with a normalization factor). The width of the distribution $|\langle C_K\rangle|^2$ can be interpreted as a correlation length between the various A modes at the output.

The distribution $\langle n_M\rangle$ is related to the distribution $\langle C_K\rangle$ through a finite Fourier transform. There is an uncertainty relation associated with every Fourier transform, which qualitatively asserts that if the $|\langle n_M\rangle|^2$ as a function of M is narrow, then the $|\langle C_K\rangle|^2$ as a function of K is wide; and vice versa. We can quantify this with the entropic uncertainty relations [19]. We first use Parseval's theorem to prove

$$\sum_M |\langle n_M\rangle|^2 = \sum_M |\langle C_M\rangle|^2 \tag{35}$$

and then define the probability distributions

$$p_K = \frac{|\langle n_K\rangle|^2}{\sum_M |\langle n_M\rangle|^2}, \quad \sigma_K = \frac{|\langle C_K\rangle|^2}{\sum_M |\langle C_M\rangle|^2}. \tag{36}$$

The p_K characterizes the distribution of a photons into the various modes, in the state $|s\rangle$. The σ_K characterizes the first-order correlations between the modes for A photons. We introduce the entropies

$$S_1 = -\sum_K p_K \ln p_K, \quad S_2 = -\sum_K \sigma_K \ln \sigma_K. \tag{37}$$

S_1 takes its minimum value zero, when $p_K = \delta(K, K_0)$, and its maximum value $\ln d$ when p_K is the uniform distribution $p_K = 1/d$. A similar remark holds for S_2. The entropic uncertainty relation states that

$$S_1 + S_2 \geqslant \ln d. \tag{38}$$

We apply this entropic uncertainty relation to the state $|s_K\rangle$ of Eq. (20) studied in Sec. III A. For $\nu \geqslant 2$ we can easily show that $\langle C_K\rangle = d^{-1/2}\nu\delta(K,0)$ which gives $\sigma_K = \delta(K,0)$ and $S_2 = 0$. For $\nu = 1$ we can show that $\langle C_K\rangle = d^{-1/2}$ which gives $\sigma_K = 1/d$ and $S_2 = \ln d$. We have seen in Eq. (21) that in this example the average number of a photons in each mode is the same; and this shows that $p_K = 1/d$ and $S_1 = \ln d$. In this example, for $\nu \geqslant 2$ the entropic inequality is satisfied as equality; and for $\nu = 1$ it becomes $S_1 + S_2 = 2\ln d > \ln d$.

We also apply the entropic uncertainty relation to the coherent state example of Eq. (27) studied in Sec. III B. We can easily show that $\langle C_K\rangle = d^{-1/2}|\beta|^2\delta(K,M)$ which gives $\sigma_K = \delta(K,M)$ and $S_2 = 0$. We also have $\langle n_K\rangle = |\beta|^2$ which gives $p_K = 1/d$ and $S_1 = \ln d$. In this example also, the entropic inequality is satisfied as equality.

The entropic inequality is a rigorous way of quantifying the uncertainty principle associated with finite Fourier transforms. At large d, for practical purposes it is intuitively more clear to use the uncertainties

$$\Delta\langle n\rangle = \left[\sum_K K^2 p_K - \left(\sum_M M p_M\right)^2\right]^{1/2},$$

$$\Delta\langle C\rangle = \left[\sum_K K^2 \sigma_K - \left(\sum_M M\sigma_M\right)^2\right]^{1/2}. \tag{39}$$

$\Delta\langle n\rangle$ is the width of the distribution of a photons into the various modes. $\Delta\langle C\rangle$ is the correlation length between the modes for A-photons.

Heuristically we can say that

$$\Delta\langle n\rangle \sim \frac{1}{\Delta\langle C\rangle}. \tag{40}$$

B. Second-order correlations

We use Eq. (32) in conjunction with Eq. (5) to prove that

$$n_M^2 = d^{-1/2}\sum_K G_K\omega^{KM}, \tag{41}$$

where

$$G_K = d^{-1/2}\sum_\Lambda C_{\Lambda-K}^\dagger C_\Lambda = d^{-3/2}\sum_{\Lambda,M,J} A_{M-\Lambda+K}A_M^\dagger A_J^\dagger A_{J-\Lambda},$$

$$[G_K,G_M] = 0, \quad G_K^\dagger = G_{-K}. \tag{42}$$

It is seen that the operators n_M^2 and G_K are related through a finite Fourier transform. Taking the expectation values with respect to an arbitrary state $|s\rangle$ we find

$$\langle n_M^2\rangle = d^{-1/2}\sum_K \langle G_K\rangle\omega^{KM}. \tag{43}$$

We then define

$$\Delta n_M^2 \equiv \langle n_M^2\rangle - \langle n_M\rangle^2,$$

$$\Delta C_K^2 \equiv \langle G_K\rangle - d^{-1/2}\sum_\Lambda \langle C_{\Lambda-K}^\dagger\rangle\langle C_\Lambda\rangle, \tag{44}$$

and using Eqs. (34) and (43), we prove that

$$\Delta n_M^2 = d^{-1/2}\sum_K \Delta C_K^2\omega^{KM}. \tag{45}$$

Δn_M^2 is the uncertainty in the number of a photons in the Mth mode, in the state $|s\rangle$. ΔC_K^2 involves second-order correlations of A photons. The finite Fourier transform implies that if the distribution ΔC_K^2 is very narrow (as a function of K), then the distribution Δn_M^2 is very wide (as a function of M), and vice versa.

We can quantify this with the entropic uncertainty relations. We first use Parseval's theorem to prove

$$\sum_M \Delta n_M^2 = \sum_K \Delta C_K^2 \tag{46}$$

and then define the probability distributions

$$p_K' = \frac{\Delta n_K^2}{\sum_M \Delta n_M^2}, \quad \sigma_K' = \frac{\Delta C_K^2}{\sum_M \Delta C_M^2} \tag{47}$$

and the entropies

$$S_1' = -\sum_K p_K'\ln p_K', \quad S_2' = -\sum_K \sigma_K'\ln\sigma_K'. \tag{48}$$

S_1' takes its minimum value zero when $p_K' = \delta(K,K_0)$; and its maximum value $\ln d$ when p_K' is the uniform distribution $p_K' = 1/d$. A similar remark holds for S_2'. The entropic uncertainty relation states that

$$S_1' + S_2' \geqslant \ln d. \tag{49}$$

We apply the entropic uncertainty relation to the coherent state example of Eq. (27) studied in Sec. III B. We can show that $\Delta C_K^2 = d^{-1/2}|\beta|^2\delta(K,0)$ which gives $\sigma_K' = \delta(K,0)$ and $S_2' = 0$. We also have $\Delta n_M^2 = |\beta|^2$ which gives $p_K' = 1/d$ and $S_1' = \ln d$. In this example the entropic inequality is satisfied as equality.

As we explained in the previous section, the entropic inequality is a rigorous way of quantifying the uncertainty principle associated with finite Fourier transforms. At large d, for practical purposes it is intuitively more clear to use the uncertainties

$$\delta(\Delta\langle n\rangle) = \left[\sum_K K^2 p_K' - \left(\sum_M M p_M'\right)^2\right]^{1/2},$$

$$\delta(\Delta\langle C\rangle) = \left[\sum_K K^2 \sigma_K' - \left(\sum_M M \sigma_M'\right)^2\right]^{1/2}. \tag{50}$$

Heuristically we can say that

$$\delta(\Delta\langle n\rangle) \sim \frac{1}{\delta(\Delta\langle C\rangle)}. \tag{51}$$

V. PHASE UNCERTAINTIES

One possible application of these devices is in multipath interferometers. In such schemes, the phase uncertainties between the output modes are of key importance. In this section, we consider how these uncertainties scale with the number of ports and the total number of atoms involved.

We denote the phase associated with the operators $\{A_M^\dagger, A_M\}$ as θ_M. The problems associated with a rigorous formalism for phases are well known [20–22]. We can express the operators $A_M^\dagger$, A_M as

$$A_M = E_M(A_M^\dagger A_M)^{1/2}, \quad A_M^\dagger = (A_M^\dagger A_M)^{1/2}E_M^\dagger,$$

$$E_M = \sum_{N=0}^{\infty} |N\rangle\langle N+1|,$$

$$E_M E_M^\dagger = \mathbf{1}, \quad E_M^\dagger E_M = \mathbf{1} - |0\rangle\langle 0|. \tag{52}$$

Here E_M is the exponential of the phase θ_M operator and it is (for infinite-dimensional Hilbert spaces) isometric but not unitary.

In a truncated D-dimensional Hilbert space, where N takes integer values modulo D, (and $|D\rangle \equiv |0\rangle$)

$$E_M = \sum_{N=0}^{D-1} |N\rangle\langle N+1|. \tag{53}$$

Now E_M is unitary and the phase operator θ_M is Hermitian. In the large-D limit we write heuristically

$$\Delta\theta_M \sim \frac{1}{\Delta N_M}. \tag{54}$$

In the example of Eq. (20) studied in Sec. III A

$$\langle N_M^2\rangle = \frac{\nu^2}{d}, \quad \langle N_M\rangle = \frac{\nu}{d}, \quad \Delta N_M^2 = \left[\frac{\nu^2}{d} - \frac{\nu^2}{d^2}\right]. \tag{55}$$

Therefore for large d

$$\Delta\theta_M \sim \frac{\sqrt{d}}{\nu}. \quad (56)$$

This phase uncertainty $\Delta\theta_M$ is of particular interest for schemes which use number correlated inputs to improve the precision of interferometers [1,6]. The scaling given by (56) is important for generalizing these schemes to multiport devices such as large arrays of condensates in optical lattices [14,15]. As an indication of the phase resolution that may be achieved by this method, experimentalists [15] have number-squeezed an array of about 12 condensates with a total number of around 10^4 atoms. For a highly squeezed system, this would enable phase resolutions of the order of $\Delta\theta_M \sim 3 \times 10^{-4}$.

VI. DISCUSSION

We have developed a theoretical analysis of multiport devices where the creation and annihilation operators at the output are related to those at the input through a finite Fourier transform. Such devices perform the unitary transformation of Eqs. (7) and (11).

We have shown that there are uncertainty relations that connect the distribution of photons into the various modes at the input, with the correlations between the various modes at the output. They have been expressed quantitatively with entropic quantities in Eqs. (38) and (49). In the large-d limit they can be written in the simpler form given in Eqs. (40) and (51).

We have presented examples which elucidate the operation of the device. In Eq. (20) we have shown that a certain superposition of number states at the input produces at the output a Schrödinger cat state comprised of number states with all photons in one mode. In Eq. (27) we have shown that coherent states with phases $2\pi M/d$ at the inputs produce a coherent state in one of the outputs and vacua in all other outputs. The reverse process can be used to split a coherent state into many other coherent states.

We have also discussed phase uncertainties in this device. We have shown that for the example of Eq. (20), $\Delta\theta_M \sim \sqrt{d}/\nu$. This result is important for considering how the measurement resolutions that can be achieved with multipath interferometers scale with the number of paths. These multiport Fourier devices may be able to be implemented in both optical and atomic systems. Possibilities include using combinations of beam splitters for photons or allowing Josephson coupling between atomic condensates trapped in an optical lattice.

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